WWW.MATEINFO.RO

REVISTA ELECTRONICA MATEINFO.RO

IUNIE 2024

ISSN 2065 - 6432

REVISTĂ DIN FEBRUARIE 2009

COORDONATOR:

ANDREI OCTAVIAN DOBRE

REDACTORI PRINCIPALI ȘI SUSȚINĂTORI PERMANENȚI AI REVISTEI

NECULAI STANCIU

MARIN CHIRCIU

ROXANA MIHAELA STANCIU

ARTICOLE

R.E.M.I. IUNIE 2024

PUTEȚI TRIMITE ARTICOLE ÎN FORMAT WORD PE REVISTA.MATEINFO@YAHOO.COM

1. H. Bergström and J. Radon triangle collaboration

By D.M. Bătineţu-Giurgiu and Neculai Stanciu

Main results I.

I.1) In any triangle ABC holds:

$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\geq\frac{27Rr}{2p};$$

I.2) In any triangle ABC holds:

$$\frac{1}{2r_a^2+5r_br_c}+\frac{1}{2r_b^2+5r_cr_a}+\frac{1}{2r_c^2+5r_ar_b}\geq\frac{9}{2(4R+r)^2+p^2};$$

I.3) In any triangle ABC with usual notations holds:

$$\frac{1}{r_a^2+2r_br_c}+\frac{1}{r_b^2+2r_cr_a}+\frac{1}{r_c^2+2r_ar_b}\geq\left(\frac{3}{4R+r}\right)^2;$$

I.4) In any triangle ABC with usual notations holds:

$$\frac{a^2}{r_br_c}+\frac{b^2}{r_cr_a}+\frac{c^2}{r_ar_b}\geq 4;$$

I.5) If $x,y,z\in R_+^*$, then in any triangle ABC with usual notations holds:

$$\frac{r_a^3}{x+yr_b+zr_c}+\frac{r_b^3}{x+yr_c+zr_a}+\frac{r_c^3}{x+yr_a+zr_b}\geq\frac{\left((4R+r)^2-2p^2\right)^2}{(4R+r)x+p^2(y+z)};$$

I.6) If $m,n\in R_+^*, m\geq n$, then in all triangle ABC holds:

$$\frac{a}{m(b+c)-na}+\frac{b}{m(c+a)-nb}+\frac{c}{m(a+b)-nc}\geq\frac{3}{2m-n};$$

I.7) If $m,n\in R_+^*$, then in all triangle ABC holds:

$$\frac{ma^2+nb^2}{a+b-c}+\frac{mb^2+nc^2}{b+c-a}+\frac{mc^2+na^2}{c+a-b}\geq 2(m+n)p;$$

I.8) If $m,n\in R_+^*$, then in all triangle ABC holds:

$$\frac{ma^2+nb^2}{(a+b-c)\cdot c}+\frac{mb^2+nc^2}{(b+c-a)\cdot a}+\frac{mc^2+na^2}{(c+a-b)\cdot b}\geq 3(m+n);$$

I.9) In any triangle ABC holds:

$$\frac{a^2}{(p-b)\cdot(p-c)}+\frac{b^2}{(p-c)\cdot(p-a)}+\frac{c^2}{(p-a)\cdot(p-b)}\geq\frac{4p^2}{(4R+r)r};$$

Proof of I.1) We have:

$$U=\sum\frac{bc}{b+c}=abc\sum\frac{1}{ab+ac},$$

and by Harald Bergström's inequality we obtain:

$$U\geq abc\cdot\frac{9}{\sum(ab+ac)}=\frac{9abc}{2(ab+bc+ca)}=\frac{9abc(a+b+c)^2}{8p^2(ab+bc+ca)}.$$

Since,

$(x+y+z)^2 \geq 3(xy+yz+zx), \forall x,y,z \in R_+$, we deduce that

$$U \geq \frac{27abc(ab+bc+ca)}{8p^2(ab+bc+ca)} = \frac{27abc}{8p^2},$$

and taking account by $abc = 4Rpr$, yields the conclusion.

Proof of I.2) By Harald Bergström's inequality we have:

$$\sum \frac{1}{2r_a^2 + 5r_b r_c} \geq \frac{9}{2\sum r_a^2 + 5\sum r_a r_b},$$

Because,

$$\sum r_a^2 = (4R+r)^2 - 2p^2 \text{ and } \sum r_a r_b = p^2,$$

follows the desired result.

Proof I.3) By Harald Bergström's inequality we have:

$$\sum \frac{1}{r_a^2 + 2r_b r_c} \geq \frac{(1+1+1)^2}{\sum r_a^2 + 2\sum r_a r_b},$$

Because

$$\sum r_a^2 = (4R+r)^2 - 2p^2 \text{ and } \sum r_a r_b = p^2$$

we obtain the desired result.

Proof of I.4) By Harald Bergström' s inequality we have:

$$\sum \frac{a^2}{r_b r_c} \geq \frac{\left(\sum a\right)^2}{\sum r_a r_b},$$

and by ,

$$\sum a = 2p \text{ and } \sum r_a r_b = p^2,$$

follows the result.

Proof of I.5) By H. Bergström's inequality we have:

$$\sum \frac{r_a^3}{x + yr_b + zr_c} = \sum \frac{r_a^4}{xr_a + yr_a r_b + zr_a r_c} \geq$$

$$\geq \frac{\left(\sum r_a^2\right)^2}{\left(x\sum r_a + (y+z)\sum r_a r_b\right)}$$

And if we taking account by,

$$\sum r_a = 4R + r,\ \sum r_a^2 = (4R+r)^2 - 2p^2 \text{ and } \sum r_a r_b = p^2,$$

we obtain the conclusion.

Proof of I.6) We have

$$U=\sum\frac{a}{m(b+c)-na}=\sum\frac{a^2}{m(ab+ac)-na^2},$$

where we apply the inequality of H. Bergström and we deduce that

$$U\geq\frac{\left(\sum a\right)^2}{m\sum(ab+ac)-n\sum a^2}=\frac{\left(\sum a\right)^2}{2m\sum ab-n\sum a^2}.$$

Since,

$$\left(\sum a\right)^2\geq 3\sum ab \text{ and } \sum a^2\geq\sum ab,$$

follows the conclusion.
Remark. For $m=1$ and $n=0$ we obtain the inequality of Nesbitt.

Proof of I.7) We have:

$$U=\sum\frac{ma^2+nb^2}{a+b-c}=m\sum\frac{a^2}{a+b-c}+n\sum\frac{b^2}{a+b-c},$$

And by H. Bergström's inequality we obtain that

$$U\geq m\cdot\frac{\left(\sum a\right)^2}{\sum(a+b-c)}+n\cdot\frac{\left(\sum b\right)^2}{\sum(a+b-c)}=(m+n)\cdot\frac{\left(\sum a\right)^2}{\sum(a+b-c)}=(m+n)\cdot\frac{(a+b+c)^2}{a+b+c}=$$
$$=2(m+n)p.$$

Proof of I.8) We have:

$$U=\sum\frac{ma^2+nb^2}{(a+b-c)\cdot c}=m\sum\frac{a^2}{ac+bc-c^2}+n\sum\frac{b^2}{ac+bc-c^2},$$

and by Bergström's inequality we get

$$U\geq m\cdot\frac{\left(\sum a\right)^2}{\sum(ac+bc)-\sum c^2}+n\cdot\frac{\left(\sum b\right)^2}{\sum(ac+bc)-\sum c^2}=$$
$$=(m+n)\cdot\frac{\left(\sum a\right)^2}{\sum(ac+bc)-\sum c^2}=(m+n)\cdot\frac{(a+b+c)^2}{2(ab+bc+ca)-(a^2+b^2+c^2)}$$

Since,

$$a^2+b^2+c^2\geq ab+bc+ca \text{ and } a^2+b^2+c^2\geq\frac{(a+b+c)^2}{3},$$

we deduce that

$$U\geq(m+n)\cdot\frac{\left(\sum a\right)^2}{2\sum a^2-\sum a^2}=(m+n)\cdot\frac{\left(\sum a\right)^2}{\sum a^2}\geq(m+n)\frac{\left(\sum a\right)^2}{\frac{\left(\sum a\right)^2}{3}}=3(m+n).$$

Proof of I.9) By Bergström's inequality we have:

$$U=\sum\frac{a^2}{(p-b)(p-c)}\geq\frac{\left(\sum a\right)^2}{\sum(p-b)(p-c)}=\frac{4p^2}{\sum(p-a)(p-b)}.$$

since,

$$\sum(p-a)(p-b)=(4R+r)r,$$

follows the conclusion.

Main results II.

II.1)If $m \in R_+$, then in any triangle ABC with usual notations holds:

$$\frac{1}{\sin^m \frac{A}{2}} + \frac{1}{\sin^m \frac{B}{2}} + \frac{1}{\sin^m \frac{C}{2}} \geq 3 \cdot 2^m;$$

II.2) If $x, y, z \in R_+^*, m \in R_+$, then in all triangle ABC holds:

$$\frac{r_a}{(xr_b + yr_c)^m} + \frac{r_b}{(xr_c + yr_a)^m} + \frac{r_c}{(xr_a + yr_b)^m} \geq \frac{(4R+r)^{m+1}}{(x+y)^m \cdot p^{2m}};$$

II.3) If $x, y, z \in R_+^*, m \in R_+$, then in all triangle ABC holds:

$$\frac{r_a}{(x + yr_b + zr_c)^m} + \frac{r_b}{(x + yr_c + zr_a)^m} + \frac{r_c}{(x + yr_a + zr_b)^m} \geq \frac{(4R+r)^{m+1}}{\left((4R+r)x + (y+z)p^2\right)^m}.$$

Proof of II.1) By J.Radon's inequality we have:

$$\sum \frac{1}{\sin^m \frac{A}{2}} \geq \frac{3^{m+1}}{\left(\sum \sin \frac{A}{2}\right)^m}$$

Since the function $f:(0,\pi) \to R_+$, $f(x) = \sin \frac{x}{2}$ with $f'(x) = \frac{1}{2}\cos\frac{x}{2}$,

$f''(x) = -\frac{1}{4}\sin\frac{x}{2} < 0, \forall x \in \left(0, \frac{\pi}{2}\right)$ is convex on $(0,\pi)$ yields that:

$$\sum \sin\frac{A}{2} \leq 3\sin\frac{A+B+C}{6} = 3\sin\frac{\pi}{6} = \frac{3}{2}.$$

So:

$$\sum \frac{1}{\sin^m \frac{A}{2}} \geq \frac{3^{m+1}}{\left(\frac{3}{2}\right)^m} = 3 \cdot 2^m.$$

Proof of II.2) From the inequality of J.Radon we have:

$$\sum \frac{r_a}{(xr_b + yr_c)^m} = \sum \frac{r_a^{m+1}}{(xr_a r_b + yr_a r_c)^m} \geq$$

$$\geq \frac{\left(\sum r_a\right)^{m+1}}{(x+y)^m \left(\sum r_a r_b\right)^m}$$

where we taking account by,

$$\sum r_a = 4R + r \text{ and } \sum r_a r_b = p^2,$$

we obtain the conclusion.

Proof of II.3) By J.Radon's inequality we have:

$$\sum \frac{r_a}{(x+yr_b+zr_c)^m} = \sum \frac{r_a^{m+1}}{(xr_a+yr_ar_b+zr_ar_c)^m} \geq$$

$$\geq \frac{\left(\sum r_a\right)^{m+1}}{\left(x\sum r_a+(y+z)\sum r_ar_b\right)^m}$$

and from well-known,

$$\sum r_a = 4R+r \text{ and } \sum r_ar_b = p^2 ,$$

we get the desired result.

Art 5800

2. Math Journal

-2-

Marin Chirciu [1]

Mathematical Journal prezintă o selecție de probleme recente din diverse publicații de specialitate .

Problema81.

If $a,b,c>0$, $abc=1$ then

$$\sum\frac{a}{3a+1}\le\frac{3}{4}.$$

Thay Hai Math, Vietnam

Soluție.

Cu substituția $(a,b,c)=\left(\frac{x}{y},\frac{y}{z},\frac{z}{x}\right)$ problema se reformulează:

If $x,y,z>0$ then

$$\sum\frac{x}{3x+y}\le\frac{3}{4}.$$

Demonstrație.

$\sum\frac{x}{3x+y}\le\frac{3}{4}\Leftrightarrow\sum\frac{y}{3x+y}\ge\frac{3}{4}$, care rezultă din inegalitatea lui Bergstrom:

$$\sum\frac{y}{3x+y}=\sum\frac{y^2}{3xy+y^2}\overset{CS}{\ge}\frac{\left(\sum y\right)^2}{\sum\left(3xy+y^2\right)}=\frac{\sum y^2+2\sum xy}{\sum y^2+3\sum xy}\overset{(1)}{\ge}\frac{3}{4},$$

unde (1) $\Leftrightarrow\frac{\sum y^2+2\sum xy}{\sum y^2+3\sum xy}\ge\frac{3}{4}\Leftrightarrow\sum y^2\ge\sum xy\Leftrightarrow\sum(x-y)^2\ge0$.

Egalitatea are loc dacă și numai dacă $x=y=z\Leftrightarrow a=b=c=1$.

Remarca.

If $a,b,c>0$, $abc=1$ and $\lambda\ge2$ then

$$\sum\frac{a}{\lambda a+1}\le\frac{3}{\lambda+1}.$$

Marin Chirciu

Soluție.

Cu substituția $(a,b,c)=\left(\frac{x}{y},\frac{y}{z},\frac{z}{x}\right)$ problema se reformulează:

If $x,y,z>0$ then

$$\sum\frac{x}{\lambda x+y}\le\frac{3}{\lambda+1}.$$

Demonstrație.

[1] *Profesor, Colegiul Național „Zinca Golescu"Pitești*

$\sum \frac{x}{\lambda x+y} \le \frac{3}{\lambda+1} \Leftrightarrow \sum \frac{y}{\lambda x+y} \ge \frac{3}{\lambda+1}$, care rezultă din inegalitatea lui Bergstrom:

$$\sum \frac{y}{\lambda x+y} = \sum \frac{y^2}{\lambda xy+y^2} \overset{CS}{\ge} \frac{\left(\sum y\right)^2}{\sum\left(\lambda xy+y^2\right)} = \frac{\sum y^2+2\sum xy}{\sum y^2+\lambda\sum xy} \overset{(1)}{\ge} \frac{3}{\lambda+1},$$

unde (1)

$$\Leftrightarrow \frac{\sum y^2+2\sum xy}{\sum y^2+\lambda\sum xy} \ge \frac{3}{\lambda+1} \Leftrightarrow (\lambda-2)\sum y^2 \ge (\lambda-2)\sum xy \Leftrightarrow (\lambda-2)\sum(x-y)^2 \ge 0.$$

Egalitatea are loc dacă și numai dacă $x=y=z \Leftrightarrow a=b=c=1$.

Problema82.

If $a,b,c \ge 0$, $a^2+b^2+c^2=1$ then

$$\sqrt{a+3}+\sqrt{b+3}+\sqrt{c+3} \ge 2+2\sqrt{3}.$$

Mathematical Inequalities 5/2024, Nguyen Minh Tho, Vietnam

Soluție.

Lema.

If $0 \le a \le 1$ then

$$\sqrt{a+3} \ge \left(2-\sqrt{3}\right)a^2+\sqrt{3}.$$

Demonstrație.

$$\sqrt{a+3} \ge \left(2-\sqrt{3}\right)a^2+\sqrt{3} \Leftrightarrow a+3 \ge \left(2-\sqrt{3}\right)^2 a^4+2\sqrt{3}\left(2-\sqrt{3}\right)^2 a^2+3 \Leftrightarrow$$

$$\Leftrightarrow a \ge \left(2-\sqrt{3}\right)^2 a^4+2\sqrt{3}\left(2-\sqrt{3}\right)^2 a^2 \Leftrightarrow a(a-1)\left[\left(2-\sqrt{3}\right)^2 a^2+\left(2-\sqrt{3}\right)a+1\right] \le 0,$$ vezi

$0 \le a \le 1$.

Folosind **Lema** obținem:

$$LHS = \sum\sqrt{a+3} \overset{Lema}{\ge} \sum\left(2-\sqrt{3}\right)a^2+3\sqrt{3} = \left(2-\sqrt{3}\right)\sum a^2+3\sqrt{3} = \left(2-\sqrt{3}\right)\cdot 1+3\sqrt{3} =$$

$$= 2+2\sqrt{3} = RHS.$$

Egalitatea are loc dacă și numai dacă $(a,b,c)=(1,0,0)$ și permutările sale.

Remarca.

Problema se poate dezvolta.

If $a,b,c \ge 0$, $a^2+b^2+c^2=1$ then

$$\sqrt{a+8}+\sqrt{b+8}+\sqrt{c+8} \ge 3+4\sqrt{2}.$$

Marin Chirciu

Soluție.

Lema.

If $0 \le a \le 1$ then

$$\sqrt{a+8} \ge \left(3-2\sqrt{2}\right)a^2+2\sqrt{2}.$$

Demonstrație.

$$\sqrt{a+8} \ge \left(3-2\sqrt{2}\right)a^2+2\sqrt{2} \Leftrightarrow a+8 \ge \left(3-2\sqrt{2}\right)^2 a^4+4\sqrt{2}\left(3-2\sqrt{2}\right)^2 a^2+8 \Leftrightarrow$$

$$\Leftrightarrow a \ge \left(3-2\sqrt{2}\right)^2 a^4+4\sqrt{2}\left(3-2\sqrt{2}\right)^2 a^2 \Leftrightarrow a(a-1)\left[\left(3-2\sqrt{2}\right)^2 a^2+\left(3-2\sqrt{2}\right)a+1\right] \le 0,$$

vezi $0 \le a \le 1$.

$$LHS = \sum\sqrt{a+8} \overset{Lema}{\ge} \sum\left(3-2\sqrt{2}\right)a^2+6\sqrt{2} = \left(3-2\sqrt{2}\right)\sum a^2+6\sqrt{2} = \left(3-2\sqrt{2}\right)\cdot 1+6\sqrt{2} =$$

$$= 3+4\sqrt{2} = RHS.$$

Egalitatea are loc dacă și numai dacă $(a,b,c)=(1,0,0)$ și permutările sale.

Problema83.

If $a,b>0$ then

$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{16}{a+b+6}\leq 3.$$

RMM 5/2024, Nguyen Hung Cuong, Vietnam

Soluție.

$$LHS=\frac{a}{a+1}+\frac{b}{b+1}+\frac{16}{a+b+6}=1-\frac{1}{a+1}+1-\frac{1}{b+1}+\frac{16}{a+b+6}=$$

$$=2-\left(\frac{1}{a+1}+\frac{1}{b+1}\right)+\frac{16}{a+b+6}\overset{CS}{\leq}2-\frac{(1+1)^2}{a+b+2}+\frac{16}{a+b+6}=2-\frac{4}{t+2}+\frac{16}{t+6}\overset{(1)}{\leq}3=RHS,$$

unde $(1)\Leftrightarrow\ 2-\frac{4}{t+2}+\frac{16}{t+6}\leq 3\ \Leftrightarrow\ (t-2)^2\geq 0$.

Egalitatea are loc dacă și numai dacă $a=b=1$.

Remarca.

If $a,b>0$ then $\lambda\geq 0$ then

$$\frac{a}{a+\lambda}+\frac{b}{b+\lambda}+\frac{4\lambda}{a+b+2\lambda}\leq 2.$$

Marin Chirciu

Soluție.

$$LHS=\frac{a}{a+\lambda}+\frac{b}{b+\lambda}+\frac{4\lambda}{a+b+2\lambda}=1-\frac{\lambda}{a+\lambda}+1-\frac{\lambda}{b+\lambda}+\frac{4\lambda}{a+b+2\lambda}=$$

$$=2-\left(\frac{\lambda}{a+\lambda}+\frac{\lambda}{b+\lambda}\right)+\frac{4\lambda}{a+b+2\lambda}\overset{CS}{\leq}2-\frac{\lambda(1+1)^2}{a+b+2\lambda}+\frac{4\lambda}{a+b+2\lambda}=2-\frac{4\lambda}{t+2\lambda}+\frac{4\lambda}{t+2\lambda}=$$

$$=2=RHS.$$

Egalitatea are loc dacă și numai dacă $a=b$.

Problema84.

If $a,b,c>0$, $abc\geq 1$ then

$$\sum\frac{a^2}{bc+1}\geq\frac{3}{2}.$$

RMM 5/2024, Nguyen Hung Cuong, Vietnam

Soluție.

$$LHS=\sum\frac{a^2}{bc+1}\overset{CS}{\geq}\frac{\left(\sum a\right)^2}{\sum(bc+1)}=\frac{\sum a^2+2\sum bc}{\sum bc+3}\overset{(1)}{\geq}\frac{3}{2}=RHS,$$

unde $(1)\Leftrightarrow\frac{\sum a^2+2\sum bc}{\sum bc+3}\geq\frac{3}{2}\ \Leftrightarrow\ 2\sum a^2+4\sum bc\geq 3\sum bc+9\Leftrightarrow 2\sum a^2+\sum bc\geq 9,$

care rezultă din $\sum a^2\geq\sum bc\geq 3$.

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Remarca.

If $a,b,c>0$, $abc\geq 1$ and $\lambda\geq 0$ then

$$\sum\frac{a^2}{bc+\lambda}\geq\frac{3}{\lambda+1}.$$

Marin Chirciu

Soluție.

$$LHS=\sum\frac{a^2}{bc+\lambda}\overset{CS}{\geq}\frac{\left(\sum a\right)^2}{\sum(bc+\lambda)}=\frac{\sum a^2+2\sum bc}{\sum bc+3\lambda}\overset{(1)}{\geq}\frac{3}{\lambda+1}=RHS,$$

unde (1) $\Leftrightarrow \frac{\sum a^2+2\sum bc}{\sum bc+3\lambda}\geq\frac{3}{\lambda+1}\Leftrightarrow(\lambda+1)\sum a^2+2(\lambda+1)\sum bc\geq 3\sum bc+9\lambda\Leftrightarrow$

$\Leftrightarrow(\lambda+1)\sum a^2+(2\lambda-1)\sum bc\geq 9\lambda$, care rezultă din $\sum a^2\geq\sum bc\geq 3$.

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Problema85.

If $x,y,z>0$, $x+y+z=1$ then

$$\sum\frac{x^2+3xy}{x+y}\leq 2.$$

Mathematics(College and High School), Amir Sofi, Kosovo

Soluție.

$$LHS=\sum\frac{x^2+3xy}{x+y}=\sum\left(x+\frac{2xy}{x+y}\right)=\sum x+\sum\frac{2xy}{x+y}\overset{AM-GM}{\leq}1+\frac{1}{2}\sum(x+y)=$$

$=1+\sum x=1+1=2=RHS$.

Egalitatea are loc dacă și numai dacă $x=y=z=\frac{1}{3}$.

Remarca.

If $x,y,z>0$, $x+y+z=1$ and $\lambda\geq 1$ then

$$\sum\frac{x^2+\lambda xy}{x+y}\leq\frac{\lambda+1}{2}.$$

Marin Chirciu

Soluție.

$$LHS=\sum\frac{x^2+\lambda xy}{x+y}=\sum\left(x+\frac{(\lambda-1)xy}{x+y}\right)=\sum x+\sum\frac{(\lambda-1)xy}{x+y}\overset{AM-GM}{\leq}1+\frac{\lambda-1}{4}\sum(x+y)=$$

$=1+\frac{\lambda-1}{2}\sum x=1+\frac{\lambda-1}{2}=\frac{\lambda+1}{2}=RHS$.

Egalitatea are loc dacă și numai dacă $x=y=z=\frac{1}{3}$.

Problema86.

E:16880. If $a,b,c>0$, $abc=1$ then

$$\sum\frac{1}{(b+c)(a+1)}\leq\frac{3}{4}.$$

E:16880, GM-4/24, Victor Ailioaiei, Vaslui

Soluție.

Lema.

If $a,b,c>0$, $abc=1$ then

$$\frac{1}{(b+c)(a+1)}\leq\frac{1}{4}.$$

Demonstrație.

$\frac{1}{(b+c)(a+1)}\overset{AM-GM}{\leq}\frac{1}{2\sqrt{bc}\cdot 2\sqrt{a}}=\frac{1}{4\sqrt{abc}}=\frac{1}{4}$, cu egalitate pentru $b=c, a=1, abc=1\Leftrightarrow$

$\Leftrightarrow a=b=c=1$.

$$LHS=\sum\frac{1}{(b+c)(a+1)}\overset{Lema}{\leq}\sum\frac{1}{4}=\frac{3}{4}=RHS.$$

Egalitatea are loc dacă şi numai dacă $a=b=c=1$.

Remarca.

If $a,b,c>0$, $abc=1$ then

$$\sum\frac{1}{(a+2)(b+c+1)}\leq\frac{1}{3}.$$

Remarca.

If $a,b,c>0$, $abc=1$ and $n\in\mathbf{N}$ then

$$\sum\frac{1}{(a+n+1)(b+c+n)}\leq\frac{3}{(n+2)^2}.$$

Marin Chirciu

Soluție.

Lema.

If $a,b,c>0$, $abc=1$ then

$$\frac{1}{(a+n+1)(b+c+n)}\leq\frac{1}{(n+2)^2}.$$

Demonstrație.

$$\frac{1}{(a+n+1)(b+c+n)}\overset{AM-GM}{\leq}\frac{1}{(n+2)\sqrt[n+2]{a\cdot\underbrace{1\cdot\ldots\cdot1}_{n+1}}\cdot(n+2)\sqrt[n+2]{b\cdot c\cdot\underbrace{1\cdot\ldots\cdot1}_{n}}}=\frac{1}{(n+2)^2\sqrt[n+2]{abc}}=$$

$=\frac{1}{(n+2)^2}$, cu egalitate pentru $a=1, b=c=1, abc=1\Leftrightarrow a=b=c=1$.

$$LHS=\sum\frac{1}{(a+n+1)(b+c+n)}\overset{Lema}{\leq}\sum\frac{1}{(n+2)^2}=\frac{3}{(n+2)^2}=RHS.$$

Egalitatea are loc dacă şi numai dacă $a=b=c=1$.

Problema87.

O662. If $a,b,c\geq0$, $a+b+c=1$ then

$$\frac{1}{8}\leq a^4+b^4+c^4+26abc\leq1.$$

Mathematical Reflections Nr.3/2024 O662. Nguyen Viet Hung, Vietnam

Solution

The first inequality.

We note $f(a,b,c)=a^4+b^4+c^4+26abc$.

We have: $f(a,b,c)-f(a,b,0)=a^4+b^4+c^4+26abc-a^4-b^4=c^4+26abc\geq0\Rightarrow$

$\Rightarrow f(a,b,c)\geq f(a,b,0)$.

$$f(a,b,0)=a^4+b^4\overset{CS}{\geq}\frac{(a+b)^4}{8}=\frac{1}{8}.$$

From $f(a,b,c)\geq f(a,b,0)$ and $f(a,b,0)\geq\frac{1}{8}\Rightarrow f(a,b,c)\geq\frac{1}{8}\Rightarrow a^4+b^4+c^4+26abc\geq\frac{1}{8}$,

equally for $(a,b,c)=\left(\frac{1}{2},\frac{1}{2},0\right)$.

The second inequality.

Lemma.

If $0\le a\le 1$ then

$$a^4\le\frac{15a^3-7a^2+a}{9}.$$

Proof.

$a^4\le\frac{15a^3-7a^2+a}{9}\Leftrightarrow 9a^4\le 15a^3-7a^2+a\Leftrightarrow a(a-1)(3a-1)^2\le 0$, see $0\le a\le 1$, equally for $a\in\left\{0,\frac{1}{3},1\right\}$.

Using **Lemma** we get:

$a^4+b^4+c^4+26abc\overset{Lema}{\le}\sum\frac{15a^3-7a^2+a}{9}+26abc\overset{(1)}{\le}1$,

where(1) $\Leftrightarrow\sum\frac{15a^3-7a^2+a}{9}+26abc\le 1\Leftrightarrow 15\sum a^3-7\sum a^2+\sum a+234abc\le 9$, which results from:

$\sum a^3=\left(\sum a\right)^3-3\sum a\sum ab+3abc=1-3\sum ab+3abc$ and

$\sum a^2=\left(\sum a\right)^2-2\sum ab=1-2\sum ab$.

It remains to show that:

$15\left(1-3\sum ab+3abc\right)-7\left(1-2\sum ab\right)+1+234abc\le 9\Leftrightarrow 31\sum ab\ge 279abc\Leftrightarrow\sum ab\ge 9abc$,

seeAM-GM: $\sum ab=\sum a\sum ab\overset{AM-GM}{\ge}3\sqrt[3]{abc}\cdot 3\sqrt[3]{a^2b^2c^2}=9abc$.

Equality occurs if and only if $a=b=c\in\left\{0,\frac{1}{3}\right\}$.

Remarca.

In ΔABC

$$\sum\frac{r}{r_a^4}+\frac{26}{rp^2}\le\frac{1}{r^3}.$$

Marin Chirciu

Soluție.

Lema.

If $x,y,z>0$, $x+y+z=1$ then

$$x^4+y^4+z^4+26xyz\le 1.$$

If $0<x<1$ then

$$x^4\le\frac{15x^3-7x^2+x}{9}.$$

Se cunoaste identitatea in triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{r_a},\frac{r}{r_b},\frac{r}{r_c}\right)$ obtinem:

$$\left(\frac{r}{r_a}\right)^4+\left(\frac{r}{r_b}\right)^4+\left(\frac{r}{r_c}\right)^4+26\frac{r}{r_a}\frac{r}{r_b}\frac{r}{r_c}\le 1\Leftrightarrow\sum\frac{r^4}{r_a^4}+26\frac{r^3}{r_ar_br_c}\le 1\Leftrightarrow\sum\frac{r}{r_a^4}+\frac{26}{rp^2}\le\frac{1}{r^3}.$$

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum \frac{r}{r_a^4} + \frac{13R}{F^2} \le \frac{1}{r^3}.$$

Marin Chirciu

Problema88.

If $a,b,c>0$, $abc\le 1$ then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 1+\frac{6}{a+b+c}.$$

Mathematics(College and High School), Amir Sofi, Kosovo

Soluție.

Avem $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \overset{AM-GM}{\ge} 3\sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}} \ge 3 \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 3$,(1).

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \overset{CS}{\ge} \frac{9}{a+b+c} \Rightarrow 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge \frac{18}{a+b+c}$,(2).

Adunând (1) și (2) $\Rightarrow 3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 3+\frac{18}{a+b+c} \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 1+\frac{6}{a+b+c}$.

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Remarca.

If $a,b,c,d>0$, $abcd\le 1$ then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\ge 1+\frac{12}{a+b+c+d}.$$

Remarca.

If $a_1,a_2,...,a_n>0$, $a_1a_2...a_n\le 1$ then

$$\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}\ge 1+\frac{n(n-1)}{a_1+a_2+...+a_n}.$$

Marin Chirciu

Soluție.

Avem $\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n} \overset{AM-GM}{\ge} n\sqrt[n]{\frac{1}{a_1}\cdot\frac{1}{a_2}\cdot...\cdot\frac{1}{a_n}} \ge n \Rightarrow \frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}\ge n$,(1).

$\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n} \overset{CS}{\ge} \frac{n^2}{a_1+a_2+...+a_n} \Rightarrow (n-1)\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}\right)\ge \frac{n^2(n-1)}{a_1+a_2+...+a_n}$,(2).

Adunând (1) și (2)

$\Rightarrow n\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}\right)\ge n+\frac{n^2(n-1)}{a_1+a_2+...+a_n} \Rightarrow \frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}\ge 1+\frac{n(n-1)}{a_1+a_2+...+a_n}$.

Egalitatea are loc dacă și numai dacă $a_1=a_2=...=a_n=1$.

Problema89.

In ΔABC

$$\sum \tan\frac{A}{2}\sqrt{\frac{1}{3}\left(\tan^2\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan^2\frac{C}{2}\right)}\ge 1.$$

RMM 5/2024,Neculai Stanciu, Buzău

Soluție.

Lema.

If $x,y>0$ then

$$\sqrt{\frac{x^2+xy+y^2}{3}}\ge\frac{x+y}{2}.$$

Demonstrație.

$\sqrt{\frac{x^2+xy+y^2}{3}}\ge\frac{x+y}{2}\Leftrightarrow(x-y)^2\ge 0$, cu egalitate pentru $x=y$.

Folosind **Lema** pentru $(x,y)=\left(\tan\frac{B}{2},\tan\frac{C}{2}\right)$ obținem:

$$LHS=\sum\tan\frac{A}{2}\sqrt{\frac{1}{3}\left(\tan^2\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan^2\frac{C}{2}\right)}\overset{Lema}{\ge}\sum\tan\frac{A}{2}\frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{2}=$$

$$=\sum\tan\frac{B}{2}\tan\frac{C}{2}=1=RHS.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1).\sum\cot\frac{A}{2}\sqrt{\frac{1}{3}\left(\cot^2\frac{B}{2}+\cot\frac{B}{2}\cot\frac{C}{2}+\cot^2\frac{C}{2}\right)}\ge 9.$$

$$2).\sum\sin\frac{A}{2}\sqrt{\frac{1}{3}\left(\sin^2\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin^2\frac{C}{2}\right)}\ge\frac{3r}{2R}.$$

$$3).\sum\cos\frac{A}{2}\sqrt{\frac{1}{3}\left(\cos^2\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\cos^2\frac{C}{2}\right)}\ge\frac{9r}{2R}.$$

$$4).\sum\sec\frac{A}{2}\sqrt{\frac{1}{3}\left(\sec^2\frac{B}{2}+\sec\frac{B}{2}\sec\frac{C}{2}+\sec^2\frac{C}{2}\right)}\ge 4.$$

$$5).\sum\csc\frac{A}{2}\sqrt{\frac{1}{3}\left(\csc^2\frac{B}{2}+\csc\frac{B}{2}\csc\frac{C}{2}+\csc^2\frac{C}{2}\right)}\ge 12.$$

Dezvoltări, Marin Chirciu

Remarca.

In ΔABC

$$\sum h_a\sqrt{\frac{1}{3}\left(h_b^2+h_bh_c+h_c^2\right)}\ge\frac{2rp^2}{R}.$$

Marin Chirciu

Soluție.

Lema.

If $x,y>0$ then

$$\sqrt{\frac{x^2+xy+y^2}{3}}\ge\frac{x+y}{2}.$$

Demonstrație.

$\sqrt{\frac{x^2+xy+y^2}{3}}\ge\frac{x+y}{2}\Leftrightarrow(x-y)^2\ge 0$, cu egalitate pentru $x=y$.

Folosind **Lema** pentru $(x,y)=(h_b,h_c)$ obținem:

$$LHS=\sum h_a\sqrt{\frac{1}{3}\left(h_b^2+h_bh_c+h_c^2\right)}\overset{Lema}{\ge}\sum h_a\frac{h_b+h_c}{2}=\sum h_bh_c=\frac{2rp^2}{R}=RHS.$$

Am folosit mai sus $\sum h_b h_c = \frac{2rp^2}{R}$.

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1).\ \sum w_a\sqrt{\frac{1}{3}\left(w_b^2+w_bw_c+w_c^2\right)}\geq 27r^2.$$

$$2).\ \sum m_a\sqrt{\frac{1}{3}\left(m_b^2+m_bm_c+m_c^2\right)}\geq p^2.$$

$$3).\ \sum s_a\sqrt{\frac{1}{3}\left(s_b^2+s_bs_c+s_c^2\right)}\geq \frac{2rp^2}{R}.$$

$$4).\ \sum r_a\sqrt{\frac{1}{3}\left(r_b^2+r_br_c+r_c^2\right)}\geq p^2.$$

Dezvoltări, Marin Chirciu

Problema90.

E:16882. If $a,b,c>0$, $a^2+b^2+c^2=3$ then

$$\sum\frac{2a^2+1}{a^4+6a^3+a^2+1}\geq 1.$$

GM-4/24, E:16882, Adrian Talaşman, Huşi

Soluție.

Lema.

If $a>0$, then

$$\frac{2a^2+1}{a^4+6a^3+a^2+1}\geq\frac{1}{2a^2+1}.$$

Demonstrație.

$$\frac{2a^2+1}{a^4+6a^3+a^2+1}\geq\frac{1}{2a^2+1}\Leftrightarrow 3a^4-6a^3+3a^2\geq 0\Leftrightarrow 3a^2\left(a-1\right)^2\geq 0.$$

$$LHS=\sum\frac{2a^2+1}{a^4+6a^3+a^2+1}\overset{Lema}{\geq}\sum\frac{1}{2a^2+1}\overset{CS}{\geq}\frac{9}{\sum\left(2a^2+1\right)}=\frac{9}{2\sum a^2+3}=\frac{9}{2\cdot 3+3}=1=RHS.$$

Egalitatea are loc dacă şi numai dacă $a=b=c=1$.

Remarca.

If $a,b,c>0$, $a^2+b^2+c^2=3$ then

$$\sum\frac{3a^2+2}{4a^4+10a^3+7a^2+4}\geq\frac{3}{5}.$$

Marin Chirciu

Soluție.

Lema.

If $a>0$, then

$$\frac{3a^2+2}{4a^4+10a^3+7a^2+4}\geq\frac{1}{3a^2+2}.$$

Demonstrație.

$$\frac{3a^2+2}{4a^4+10a^3+7a^2+4}\geq\frac{1}{3a^2+2}\Leftrightarrow 5a^4-10a^3+5a^2\geq 0\Leftrightarrow 5a^2\left(a-1\right)^2\geq 0.$$

$$\sum \frac{3a^2+2}{4a^4+10a^3+7a^2+4} \overset{Lema}{\geq} \sum \frac{1}{3a^2+2} \overset{CS}{\geq} \frac{9}{\sum(3a^2+2)} = \frac{9}{3\sum a^2+6} = \frac{9}{3\cdot 3+6} = \frac{3}{5}.$$

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Problema91.

S:L.24:128. If $x, y > 0$ then

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{x^2+y^2} \geq \frac{10}{(x+y)^2}.$$

SGM-4/24, S:24.128, Costel Anghel, Negreni, Olt

Soluție.

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{x^2+y^2} \geq \frac{10}{(x+y)^2} \Leftrightarrow x^6+2x^5y-6x^4y^2+6x^3y^3-6x^2y^4+2xy^5+y^6 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (x-y)^2\left(x^4+4x^3y+x^2y^2+4xy^3+y^4\right) \geq 0$$

Egalitatea are loc dacă și numai dacă $x=y$.

Remarca.

If $x, y > 0$ and $\lambda \leq 12$ then

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{\lambda}{x^2+y^2} \geq \frac{2(\lambda+4)}{(x+y)^2}.$$

Marin Chirciu

Soluție.

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{\lambda}{x^2+y^2} \geq \frac{2(\lambda+4)}{(x+y)^2} \Leftrightarrow$$

$$x^6+2x^5y-(\lambda+5)x^4y^2+2(\lambda+2)x^3y^3-(\lambda+5)x^2y^4+2xy^5+y^6 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (x-y)^2\left(x^4+4x^3y+(2-\lambda)x^2y^2+4xy^3+y^4\right) \geq 0$, care rezultă din:

$(x-y)^2 \geq 0$ și $\left(x^4+4x^3y+(2-\lambda)x^2y^2+4xy^3+y^4\right) \geq 0$, vezi

$$x^4+4x^3y+(2-\lambda)x^2y^2+4xy^3+y^4 \overset{\lambda\leq 12}{\geq} x^4+4x^3y-10x^2y^2+4xy^3+y^4 =$$

$$=(x-y)^2\left(x^2+6xy+y^2\right) \geq 0.$$

Egalitatea are loc dacă și numai dacă $x=y$.

Remarca.

If $x, y > 0$ then

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{12}{x^2+y^2} \geq \frac{32}{(x+y)^2}.$$

Marin Chirciu

Problema92.

If $x, y, z > 0$, $x+y+z=1$ then

$$\sum \frac{x^2+y}{y+z} \geq 2.$$

Mathematical Inequalities 5/2024, Dorin Marghidanu, Corabia, Olt

Soluție.

$$LHS = \sum \frac{x^2+y}{y+z} = \sum \frac{x(1-y-z)+y}{y+z} = \sum \frac{-x(y+z)+x+y}{y+z} = \sum\left(-x+\frac{x+y}{y+z}\right) =$$

$$=-\sum x+\sum\frac{x+y}{y+z}\overset{AM-GM}{\geq}-1+3\sqrt[3]{\prod\frac{x+y}{y+z}}=-1+3=2=RHS.$$

Egalitatea are loc dacă și numai dacă $x=y=z=\frac{1}{3}$.

Remarca.

In ΔABC

$$\sum\frac{\left(h_a^2+rh_b\right)h_c}{h_a^2\left(h_b+h_c\right)}\geq 2.$$

Soluție.

Lema.

If $x,y,z>0$, $x+y+z=1$ then

$$\sum\frac{x^2+y}{y+z}\geq 2.$$

Se cunoaste identitatea in triunghi $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}\Leftrightarrow\frac{r}{h_a}+\frac{r}{h_b}+\frac{r}{h_c}=1$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{r}{h_a},\frac{r}{h_b},\frac{r}{h_c}\right)$ obtinem:

$$\sum\frac{\left(\frac{r}{h_a}\right)^2+\frac{r}{h_b}}{\frac{r}{h_b}+\frac{r}{h_c}}\geq 2\Leftrightarrow\sum\frac{\left(h_a^2+rh_b\right)h_c}{h_a^2\left(h_b+h_c\right)}\geq 2.$$

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum\frac{\left(r_a^2+rr_b\right)r_c}{r_a^2\left(r_b+r_c\right)}\geq 2.$$

Marin Chirciu

Problema93.

If $x,y,z>0$, $x+y+z=3$ then

$$\sum\frac{y^2+z^2}{1+x^2}\geq 3.$$

Mathematical Inequalities 5/2024, Witcher Ben

Soluție.

$$LHS=\sum\frac{y^2+z^2}{1+x^2}=\sum\frac{y^2}{1+x^2}+\sum\frac{z^2}{1+x^2}=\sum\frac{y^4}{y^2+x^2y^2}+\sum\frac{z^4}{z^2+z^2x^2}\overset{CS}{\geq}\frac{\left(\sum y^2\right)^2}{\sum\left(y^2+x^2y^2\right)}+$$

$$+\frac{\left(\sum z^2\right)^2}{\sum\left(z^2+z^2x^2\right)}=\frac{2\left(\sum x^2\right)^2}{\sum x^2+\sum x^2y^2}\overset{(1)}{\geq}3=RHS,$$

unde (1) $\Leftrightarrow\frac{2\left(\sum x^2\right)^2}{\sum x^2+\sum x^2y^2}\geq 3\Leftrightarrow 2\left(\sum x^2\right)^2\geq 3\sum x^2+3\sum x^2y^2$, care rezultă din:

$\left(\sum x^2\right)^2\geq 3\sum x^2$ și $\left(\sum x^2\right)^2\geq 3\sum x^2y^2$.

Să demonstrăm $\left(\sum x^2\right)^2 \geq 3\sum x^2$.

Avem $\left(\sum x^2\right)^2 \geq 3\sum x^2 \Leftrightarrow \sum x^2 \geq 3$, vezi $\sum x^2 \overset{CS}{\geq} \frac{\left(\sum x\right)^2}{3} = \frac{3^2}{3} = 3$

$\Rightarrow 2\sum x^4 \geq \sum x\left(y^3 + z^3\right)$.

Să demonstrăm $\left(\sum x^2\right)^2 \geq 3\sum x^2 y^2$.

Folosim $\left(\sum a\right)^2 \geq 3\sum bc$ pentru $(a,b,c) = \left(x^2, y^2, z^2\right)$.

Egalitatea are loc dacă și numai dacă $x = y = z = 1$.

Remarca.

In ΔABC

$$\sum \frac{\frac{1}{r_b^2} + \frac{1}{r_c^2}}{1 + \frac{9r^2}{r_a^2}} \geq \frac{1}{3r^2}.$$

Soluție.

Lema.

If $x, y, z > 0$, $x + y + z = 3$ then

$$\sum \frac{y^2 + z^2}{1 + x^2} \geq 3.$$

Se cunoaste identitatea in triunghi $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r} \Leftrightarrow \frac{3r}{r_a} + \frac{3r}{r_b} + \frac{3r}{r_c} = 3$.

Folosind **Lema** pentru $(x, y, z) = \left(\frac{3r}{r_a}, \frac{3r}{r_b}, \frac{3r}{r_c}\right)$ obtinem:

$$\sum \frac{\left(\frac{3r}{r_b}\right)^2 + \left(\frac{3r}{r_c}\right)^2}{1 + \left(\frac{3r}{r_a}\right)^2} \geq 3 \Leftrightarrow \sum \frac{\frac{1}{r_b^2} + \frac{1}{r_c^2}}{1 + \frac{9r^2}{r_a^2}} \geq \frac{1}{3r^2}.$$

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum \frac{\frac{1}{h_b^2} + \frac{1}{h_c^2}}{1 + \frac{9r^2}{h_a^2}} \geq \frac{1}{3r^2}.$$

Marin Chirciu

Problema94.

In ΔABC

$$\sum \frac{m_a}{m_b + m_c} \cot \frac{A}{2} \geq \frac{3\sqrt{3}}{2}.$$

RMM 5/2024,Nguyen Hung Cuong, Vietnam

Soluție.

Tripletele $\left(\frac{m_a}{m_b+m_c}, \frac{m_b}{m_c+m_a}, \frac{m_c}{m_a+m_b}\right)$ și $\left(\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}\right)$ sunt la fel ordonate.

Folosind inegalitatea lui Chebyshev obținem:

$$LHS = \sum \frac{m_a}{m_b+m_c}\cot\frac{A}{2} \overset{Chebyshev}{\geq} \frac{1}{3}\sum\frac{m_a}{m_b+m_c}\sum\cot\frac{A}{2} \overset{Nesbitt}{\geq} \frac{1}{3}\cdot\frac{3}{2}\cdot\frac{p}{r} \overset{Mitrinovic}{\geq} \frac{1}{2}\cdot\frac{3\sqrt{3}r}{r} =$$

$$= \frac{3\sqrt{3}}{2} = RHS.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1).\ \sum \frac{h_a}{h_b+h_c}\cot\frac{A}{2} \geq \frac{3\sqrt{3}}{2}.$$

$$2).\ \sum \frac{w_a}{w_b+w_c}\cot\frac{A}{2} \geq \frac{3\sqrt{3}}{2}.$$

$$3).\ \sum \frac{s_a}{s_b+s_c}\cot\frac{A}{2} \geq \frac{3\sqrt{3}}{2}.$$

$$4).\ \sum \frac{h_a}{h_b+h_c}\cos\frac{A}{2} \geq \frac{3\sqrt{3}r}{2R}.$$

$$5).\ \sum \frac{w_a}{w_b+w_c}\cos\frac{A}{2} \geq \frac{3\sqrt{3}r}{2R}.$$

$$6).\ \sum \frac{m_a}{m_b+m_c}\cos\frac{A}{2} \geq \frac{3\sqrt{3}r}{2R}.$$

$$7).\ \sum \frac{s_a}{s_b+s_c}\cos\frac{A}{2} \geq \frac{3\sqrt{3}r}{2R}.$$

Dezvoltări, Marin Chirciu

Problema95.

In ΔABC

$$\sum \frac{r_a}{r_b+r_c}\tan\frac{A}{2} \geq \frac{\sqrt{3}}{2}.$$

Soluție.

Tripletele $\left(\frac{r_a}{r_b+r_c}, \frac{r_b}{r_c+r_a}, \frac{r_c}{r_a+r_b}\right)$ și $\left(\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}\right)$ sunt la fel ordonate.

$$LHS = \sum \frac{r_a}{r_b+r_c}\cot\frac{A}{2} \overset{Chebyshev}{\geq} \frac{1}{3}\sum\frac{r_a}{r_b+r_c}\sum\tan\frac{A}{2} \overset{Nesbitt}{\geq} \frac{1}{3}\cdot\frac{3}{2}\cdot\frac{4R+r}{p} \overset{Doucet}{\geq} \frac{1}{2}\cdot\sqrt{3} = \frac{\sqrt{3}}{2} = RHS.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

În aceeași clasă de probleme.

In ΔABC

$$\sum \frac{r_a}{r_b+r_c}\sin\frac{A}{2} \geq \frac{1}{2}\left(1+\frac{r}{R}\right).$$

Ine Math ,Marin Chirciu

Soluție.

Tripletele $\left(\frac{r_a}{r_b+r_c},\frac{r_b}{r_c+r_a},\frac{r_c}{r_a+r_b}\right)$ şi $\left(\sin\frac{A}{2},\sin\frac{B}{2},\sin\frac{C}{2}\right)$ sunt la fel ordonate.

$$LHS=\sum\frac{r_a}{r_b+r_c}\sin\frac{A}{2}\overset{Chebyshev}{\geq}\frac{1}{3}\sum\frac{r_a}{r_b+r_c}\sum\sin\frac{A}{2}\overset{Nesbitt}{\geq}\frac{1}{3}\cdot\frac{3}{2}\cdot\left(1+\frac{r}{R}\right)=\frac{1}{2}\left(1+\frac{r}{R}\right)=RHS.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Problema96.

In ΔABC

$$\sum\frac{a}{b+c}(h_b+h_c)\geq 9r.$$

RMM 5/2024,Zaza Mzhavanadze, Georgia

Soluție.

Lema.

In ΔABC

$$\sum\frac{a}{b+c}(h_b+h_c)=\frac{p^2-r^2-4Rr}{R}.$$

$$LHS=\sum\frac{a}{b+c}(h_b+h_c)=\frac{p^2-r^2-4Rr}{R}\overset{Gerretsen}{\geq}\frac{16Rr-5r^2-r^2-4Rr}{R}=\frac{12Rr-6r^2}{R}=$$

$$=\frac{6r(2R-r)}{R}\overset{Euler}{\geq}9r=RHS.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$6r\left(2-\frac{r}{R}\right)\leq\sum\frac{a}{b+c}(h_b+h_c)\leq\frac{9R}{2}.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\sum\frac{a}{b+c}(h_b+h_c)=\frac{p^2-r^2-4Rr}{R}.$$

Inegalitatea din dreapta.

$$\sum\frac{a}{b+c}(h_b+h_c)=\frac{p^2-r^2-4Rr}{R}\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2-r^2-4Rr}{R}=\frac{4R^2+2r^2}{R}\overset{Euler}{\leq}\frac{9R}{2}.$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Inegalitatea din stânga.

$$\sum\frac{a}{b+c}(h_b+h_c)=\frac{p^2-r^2-4Rr}{R}\overset{Gerretsen}{\geq}\frac{16Rr-5r^2-r^2-4Rr}{R}=\frac{12Rr-6r^2}{R}=$$

$$=\frac{6r(2R-r)}{R}=6r\left(2-\frac{r}{R}\right).$$

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$9r\leq 6r\left(2-\frac{r}{R}\right)\leq\sum\frac{a}{b+c}(h_b+h_c)\leq\frac{9R}{2}.$$

Problema97.

UP.548. Find

$$\Omega=\int_1^e \frac{1-\ln x}{x^2+\ln^2 x}dx.$$

RMM- Number 37, Summer 2025, Daniel Sitaru, Romania

Solutie.

$$\Omega=\int_1^e \frac{1-\ln x}{x^2+\ln^2 x}dx=\int_1^e \frac{\frac{1-\ln x}{\ln^2 x}}{\left(\frac{x}{\ln x}\right)^2+1}dx=\int_1^e \frac{-\left(\frac{x}{\ln x}\right)'}{\left(\frac{x}{\ln x}\right)^2+1}dx=-\arctan\frac{x}{\ln x}\Big|_1^e=$$

$$=-\left(\arctan\frac{e}{\ln e}-\arctan\frac{1}{\ln 1}\right)=-\left(\arctan e-\frac{\pi}{2}\right)=\frac{\pi}{2}-\arctan e.$$

Remarca.

Find

$$\Omega=\int_1^e \frac{1-\ln x}{x^2-\ln^2 x}dx.$$

Marin Chirciu

Solutie.

$$\Omega=\int_1^e \frac{1-\ln x}{x^2-\ln^2 x}dx=\int_1^e \frac{\frac{1-\ln x}{\ln^2 x}}{\left(\frac{x}{\ln x}\right)^2-1}dx=\int_1^e \frac{-\left(\frac{x}{\ln x}\right)'}{\left(\frac{x}{\ln x}\right)^2-1}dx=-\frac{1}{2}\ln\left|\frac{\frac{x}{\ln x}-1}{\frac{x}{\ln x}+1}\right|\Bigg|_1^e=-\frac{1}{2}\ln\left|\frac{x-\ln x}{x+\ln x}\right|\Bigg|_1^e=$$

$$=-\frac{1}{2}\ln\left|\frac{x-\ln x}{x+\ln x}\right|\Bigg|_1^e=-\frac{1}{2}\left(\ln\frac{e-\ln e}{e+\ln e}-\ln\frac{1-\ln 1}{1+\ln 1}\right)=-\frac{1}{2}\left(\ln\frac{e-1}{e+1}-\ln 1\right)=-\frac{1}{2}\ln\frac{e-1}{e+1}=\frac{1}{2}\ln\frac{e+1}{e-1}.$$

Problema98.

In ΔABC

$$9\left(\frac{R}{r}\right)^{-1}\le\sum\frac{2r_a+h_a}{2r_a}\le\frac{9}{4}\left(\frac{R}{r}\right).$$

Mathematical Inequalities, Problem(381), Kostas Geronikolas, Greece

Soluție.

Lema.

In ΔABC

$$\sum\frac{2r_a+h_a}{2r_a}=\frac{p^2+r^2+4Rr}{4Rr}.$$

Inegalitatea din dreapta.

$$\sum\frac{2r_a+h_a}{2r_a}=\frac{p^2+r^2+4Rr}{4Rr}\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2+r^2+4Rr}{4Rr}=\frac{R^2+2Rr+r^2}{Rr}\overset{Euler}{\le}\frac{9R}{4r}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Inegalitatea din stânga.

$$\sum\frac{2r_a+h_a}{2r_a}=\frac{p^2+r^2+4Rr}{4Rr}\overset{Gerretsen}{\ge}\frac{16Rr-5r^2+r^2+4Rr}{4Rr}=\frac{5R-r}{R}\overset{Euler}{\ge}\frac{9r}{R}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$3\left(\frac{2r}{R}\right)\le\sum\frac{h_a}{r_a}\le3\left(\frac{R}{r}\right)^2.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\sum\frac{h_a}{r_a}=\frac{p^2+r^2-8Rr}{2Rr}.$$

Inegalitatea din dreapta.

$$\sum\frac{h_a}{r_a}=\frac{p^2+r^2-8Rr}{2Rr}\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2+r^2-8Rr}{2Rr}=\frac{2\left(R^2-Rr+r^2\right)}{Rr}\overset{Euler}{\le}\frac{3R^2}{4r^2}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Inegalitatea din stânga.

$$\sum\frac{h_a}{r_a}=\frac{p^2+r^2-8Rr}{2Rr}\overset{Gerretsen}{\ge}\frac{16Rr-5r^2+r^2-8Rr}{2Rr}=\frac{2(2R-r)}{R}\overset{Euler}{\ge}\frac{6r}{R}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$3\le\sum\frac{r_a}{h_a}\le\frac{2R}{r}-1.$$

Marin Chirciu

Remarca.

In ΔABC

$$\sum\frac{h_a}{r_a}\le\sum\frac{r_a}{h_a}.$$

Marin Chirciu

Soluție.

Folosind **Lemele** de mai sus, inegalitatea se scrie:

$\frac{p^2+r^2-8Rr}{2Rr}\le\frac{2R-r}{r}\Leftrightarrow p^2\le4R^2+6Rr-r^2$, care rezultă din inegalitatea lui Gerretsen $p^2\le4R^2+4Rr+3r^2$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema99.

If $x,y,z>0$, $x+y+z=3$ then

$$\sum\frac{x^3}{y^2+z^2}\ge\frac{3}{2}.$$

Mathematical Inequalities 5/2024, Witcher Ben

Soluție.

$$\sum\frac{x^3}{y^2+z^2}=\sum\frac{3x^3}{(x+y+z)\left(y^2+z^2\right)}=\sum\frac{3x^4}{x(x+y+z)\left(y^2+z^2\right)}\overset{CS}{\ge}\frac{3\left(\sum x^2\right)^2}{\sum x(x+y+z)\left(y^2+z^2\right)}=$$

$$=\frac{3\left(\sum x^4+2\sum x^2y^2\right)}{2\sum x^2y^2+\sum x\left(y^3+z^3\right)+2xyz\sum x}\overset{(1)}{\ge}\frac{3}{2}=RHS,$$

unde (1) $\Leftrightarrow \dfrac{3\left(\sum x^4+2\sum x^2y^2\right)}{2\sum x^2y^2+\sum x\left(y^3+z^3\right)+2xyz\sum x} \ge \dfrac{3}{2} \Leftrightarrow$

$\Leftrightarrow 2\sum x^4+4\sum x^2y^2 \ge 2\sum x^2y^2+\sum x\left(y^3+z^3\right)+2xyz\sum x \Leftrightarrow$

$\Leftrightarrow 2\sum x^4+2\sum x^2y^2 \ge \sum x\left(y^3+z^3\right)+2xyz\sum x$, care rezultă din:

$2\sum x^4 \ge \sum x\left(y^3+z^3\right)$ și $\sum x^2y^2 \ge xyz\sum x$.

Să demonstrăm $2\sum x^4 \ge \sum x\left(y^3+z^3\right)$.

Avem $\left(x^4+3y^4\right)+\left(x^4+3z^4\right) \overset{AM-GM}{\ge} 4\sqrt[4]{x^4y^4y^4y^4}+4\sqrt[4]{x^4z^4z^4z^4}=4xy^3+4xz^3=4x\left(y^3+z^3\right) \Rightarrow$

$\Rightarrow 2\sum x^4 \ge \sum x\left(y^3+z^3\right)$.

Să demonstrăm $\sum x^2y^2 \ge xyz\sum x$.

Folosim $\sum a^2 \ge \sum bc$ pentru $(a,b,c)=(xy,yz,zx)$.

Egalitatea are loc dacă și numai dacă $x=y=z=1$.

Remarca.

In ΔABC

$$\sum \frac{1}{r_a^3\left(\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)} \ge \frac{1}{2r}.$$

Soluție.

Lema.

If $x,y,z>0$, $x+y+z=3$ then

$$\sum \frac{x^3}{y^2+z^2} \ge \frac{3}{2}.$$

Se cunoaste identitatea in triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r} \Leftrightarrow \frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ obtinem:

$$\sum \frac{\left(\frac{3r}{r_a}\right)^3}{\left(\frac{3r}{r_b}\right)^2+\left(\frac{3r}{r_c}\right)^2} \ge \frac{3}{2} \Leftrightarrow \sum \frac{1}{r_a^3\left(\frac{1}{r_b^2}+\frac{1}{r_c^2}\right)} \ge \frac{1}{2r}.$$

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Remarca.

Problema se poate dezvolta.

In ΔABC

$$\sum \frac{1}{h_a^3\left(\frac{1}{h_b^2}+\frac{1}{h_c^2}\right)} \ge \frac{1}{2r}.$$

Marin Chirciu

Problema100.

If $a,b,c>0$, $a^2+b^2+c^2 \le 3$ then

$$\sum \frac{1}{4-a} \le 1.$$

Mathematical Inequalities 5/2024

Soluție.

Lema.

If $0 < a < 2$,then

$$\frac{1}{4-a} \le \frac{a^2+5}{18}.$$

Demonstrație.

$\frac{1}{4-a} \le \frac{a^2+5}{18} \Leftrightarrow a^3 - 4a^2 + 5a - 2 \le 0 \Leftrightarrow (a-1)^2(a-2) \le 0$.

$LHS = \sum \frac{1}{4-a} \overset{Lema}{\le} \sum \frac{a^2+5}{18} = \frac{\sum a^2 + 15}{18} \le \frac{3+15}{18} = 1 = RHS$.

Am folosit $a^2+b^2+c^2 \le 3 \Rightarrow 3-a^2 > 0 \Rightarrow a < \sqrt{3} \Rightarrow a < \sqrt{3} < 2 < 4$.

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Remarca.

If $a,b,c > 0$, $a^2+b^2+c^2 = 3$ and $\lambda \ge 2+\sqrt{3}$ then

$$\sum \frac{1}{\lambda - a} \le \frac{3}{\lambda - 1}.$$

Marin Chirciu

Soluție.

Lema.

If $0 < a < \lambda - 2$ then

$$\frac{1}{\lambda - a} \le \frac{a^2 + 2\lambda - 3}{2(\lambda-1)^2}.$$

Demonstrație.

$\frac{1}{\lambda - a} \le \frac{a^2 + 2\lambda - 3}{2(\lambda-1)^2} \Leftrightarrow a^3 - \lambda a^2 + (2\lambda - 3)a + 2 - \lambda \le 0 \Leftrightarrow (a-1)^2(a+2-\lambda) \le 0$,

vezi $0 < a < \lambda - 2$.

$LHS = \sum \frac{1}{\lambda - a} \overset{Lema}{\le} \sum \frac{a^2+2\lambda-3}{2(\lambda-1)^2} = \frac{\sum a^2 + 3(2\lambda-3)}{2(\lambda-1)^2} \le \frac{3+3(2\lambda-3)}{2(\lambda-1)^2} = \frac{3}{\lambda-1} = RHS$.

Am folosit $a^2+b^2+c^2 = 3 \Rightarrow 3-a^2 > 0 \Rightarrow a < \sqrt{3} \Rightarrow a < \sqrt{3} \le \lambda - 2 < \lambda$, vezi $\lambda \ge 2+\sqrt{3}$.

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Problema101.

If $a,b,c > 0$ then

$$\sum \frac{a}{\sqrt{2a+b}} \le \sqrt{a+b+c}.$$

Mathematical Inequalities, Witcher Ben

Soluție

$LHS = \sum \frac{a}{\sqrt{2a+b}} \overset{CBS}{\le} \sqrt{\sum a \sum \frac{a}{2a+b}} \overset{(1)}{\le} \sqrt{\sum a \cdot 1} = \sqrt{a+b+c} = RHS$,

unde (1) $\Leftrightarrow \sum \frac{a}{2a+b} \le 1 \Leftrightarrow \sum \frac{b}{2a+b} \ge 1$, care rezultă din inegalitatea lui Bergstrom:

$$\sum\frac{b}{2a+b}=\sum\frac{b^2}{2ab+b^2}\overset{CS}{\geq}\frac{\left(\sum b\right)^2}{\sum\left(2ab+b^2\right)}=\frac{\sum a^2+2\sum ab}{\sum a^2+2\sum ab}=1.$$

Egalitatea are loc dacă și numai dacă $a=b=c$.

Remarca.

If $a,b,c>0$ and $\lambda\geq 2$ then

$$\sum\frac{a}{\sqrt{\lambda a+b}}\leq\sqrt{\frac{3(a+b+c)}{\lambda+1}}.$$

Marin Chirciu

Soluție

$$LHS=\sum\frac{a}{\sqrt{\lambda a+b}}\overset{CBS}{\leq}\sqrt{\sum a\sum\frac{a}{\lambda a+b}}\overset{(1)}{\leq}\sqrt{\sum a\cdot\frac{3}{\lambda+1}}=\sqrt{\frac{3(a+b+c)}{\lambda+1}}=RHS,$$

unde (1) $\Leftrightarrow \sum\frac{a}{\lambda a+b}\leq\frac{3}{\lambda+1}\Leftrightarrow\sum\frac{b}{\lambda a+b}\geq\frac{3}{\lambda+1}$, vezi inegalitatea lui Bergstrom:

$$\sum\frac{b}{\lambda a+b}=\sum\frac{b^2}{\lambda ab+b^2}\overset{CS}{\geq}\frac{\left(\sum b\right)^2}{\sum\left(\lambda ab+b^2\right)}=\frac{\sum a^2+2\sum ab}{\sum a^2+\lambda\sum ab}\overset{(1)}{\geq}\frac{3}{\lambda+1}.$$

Egalitatea are loc dacă și numai dacă $a=b=c$.

Problema102.

If $a,b>0$ then

$$\frac{a^5}{b}+\frac{b^5}{a}+4\left(\frac{1}{a}+\frac{1}{b}\right)\geq 10.$$

RMM 5/2024,Nguyen Hung Cuong, Vietnam

Soluție

$$LHS=\frac{a^5}{b}+\frac{b^5}{a}+4\left(\frac{1}{a}+\frac{1}{b}\right)\overset{Holder}{\geq}\frac{(a+b)^5}{2^3(b+a)}+4\frac{(1+1)^2}{a+b}=\frac{t^4}{8}+\frac{16}{t}\overset{(1)}{\geq}10=RHS,$$

unde (1) $\Leftrightarrow \frac{t^4}{8}+\frac{16}{t}\geq 10\Leftrightarrow t^5-80t+128\geq 0\Leftrightarrow (t-2)^2\left(t^3+4t^2+12t+32\right)\geq 0$.

Egalitatea are loc dacă și numai dacă $a=b=1$.

Remarca.

If $a,b>0$ and $\lambda\geq 0$, $n\in\mathbf{N}^*$ then

$$\frac{a^{2n+1}}{b}+\frac{b^{2n+1}}{a}+\lambda\left(\frac{1}{a}+\frac{1}{b}\right)\geq 2(\lambda+1).$$

Marin Chirciu

Soluție

$$LHS=\frac{a^{2n+1}}{b}+\frac{b^{2n+1}}{a}+\lambda\left(\frac{1}{a}+\frac{1}{b}\right)\overset{Holder}{\geq}\frac{(a+b)^{2n+1}}{2^{2n-1}(b+a)}+\lambda\frac{(1+1)^2}{a+b}=\frac{t^{2n}}{2^{2n-1}}+\frac{4\lambda}{t}\overset{(1)}{\geq}2(\lambda+1)=RHS,$$

unde (1) $\Leftrightarrow \frac{t^{2n}}{2^{2n-1}}+\frac{4\lambda}{t}\overset{(1)}{\geq}2(\lambda+1)\Leftrightarrow t^{2n+1}-2^{2n}(\lambda+1)t+2^{2n+1}\lambda\geq 0\Leftrightarrow$

$\Leftrightarrow (t-2)^2\left(t^{2n-1}+4t^{2n-2}+12t^{2n-3}+\ldots+2^{2n-1}\lambda\right)\geq 0$.

Egalitatea are loc dacă și numai dacă $a=b=1$.

Problema103.

O666. In ΔABC

a) $AI^2+BI^2+CI^2\geq\frac{a^2+b^2+c^2}{3}$.

b) $AI^2+BI^2+CI^2\geq\sum\frac{a^2}{2+\cos B+\cos C}$.

Mathematical Reflections, 3/2024, Nguyen Viet Hung, Vietnam

Solution

a)Using $AI=\frac{r}{\sin\frac{A}{2}}$ we have:

$$\sum AI^2=\sum\frac{r^2}{\sin^2\frac{A}{2}}=r^2\sum\frac{1}{\sin^2\frac{A}{2}}=r^2\cdot\frac{p^2+r^2-8Rr}{r^2}=p^2+r^2-8Rr\overset{(1)}{\geq}\frac{2\left(p^2-r^2-4Rr\right)}{3}=$$
$$=\frac{a^2+b^2+c^2}{3},$$

(1) $\Leftrightarrow$ $p^2+r^2-8Rr\geq\frac{2\left(p^2-r^2-4Rr\right)}{3}\Leftrightarrow p^2\geq 16Rr-5r^2$,(Gerretsen).

Equality occurs if and only if triangle is equilateral.

b) Using $AI=\frac{r}{\sin\frac{A}{2}}$ and $r=(p-a)\tan\frac{A}{2}$ we have:

$$\sum AI^2=\sum\frac{r^2}{\sin^2\frac{A}{2}}=\sum\frac{(p-a)^2\tan^2\frac{A}{2}}{\sin^2\frac{A}{2}}=\sum\frac{(p-a)^2}{\cos^2\frac{A}{2}}=\sum\frac{(p-a)^2}{\frac{1+\cos A}{2}}=\sum\frac{2(p-a)^2}{1+\cos A}=$$
$$=\sum\left(\frac{(p-b)^2}{1+\cos B}+\frac{(p-c)^2}{1+\cos C}\right)\overset{CS}{\geq}\sum\left(\frac{(p-b+p-c)^2}{1+\cos B+1+\cos C}\right)=\sum\frac{a^2}{2+\cos B+\cos C}.$$

Equality occurs if and only if triangle is equilateral.

Problema104.

Solve in real numbers

$$\sqrt{4x^2+5x+1}-2\sqrt{x^2-x+1}=9x-3.$$

THCS 5/2024,Tran Anh Tuan, Vietnam

Soluție.

Notăm $\begin{cases}a=\sqrt{4x^2+5x+1}\\ b=\sqrt{x^2-x+1}\end{cases}\Rightarrow\begin{cases}a-2b=9x-3\\ a^2-4b^2=9x-3\end{cases}\Rightarrow a^2-4b^2=a-2b\Leftrightarrow$

$\Leftrightarrow(a-2b)(a+2b-1)=0$.

Cazul1).Dacă $a=2b\Rightarrow\sqrt{4x^2+5x+1}=2\sqrt{x^2-x+1}\Rightarrow x=\frac{1}{3}$, care convine.

Cazul2).Dacă $a=1-2b\Rightarrow\sqrt{4x^2+5x+1}=1-2\sqrt{x^2-x+1}$, care nu convine, deoarece $1-2\sqrt{x^2-x+1}<0\Leftrightarrow 2\sqrt{x^2-x+1}>1\Leftrightarrow 4x^2-4x+3>0$, vezi $\Delta<0$.

Deducem că ecuația are singura soluție reală $x=\frac{1}{3}$.

Remarca.

Solve in real numbers

$$\sqrt{9x^2+7x+1}-3\sqrt{x^2-x+1}=16x-8.$$

Soluție.

Deducem că ecuația are singura soluție reală $x=\frac{1}{2}$.

Remarca.

Let $\lambda>\frac{2}{\sqrt{3}}$ fixed. Solve in real numbers

$$\sqrt{\lambda^2x^2+\left(\lambda^2-2\right)x+1}-\lambda\sqrt{x^2-x+1}=2\left(\lambda^2-1\right)x+1-\lambda^2.$$

Marin Chirciu

Soluție.

Notăm $\begin{cases} a=\sqrt{\lambda^2x^2+\left(\lambda^2-2\right)x+1} \\ b=\sqrt{x^2-x+1} \end{cases} \Rightarrow \begin{cases} a-\lambda b=2\left(\lambda^2-1\right)x+1-\lambda^2 \\ a^2-\lambda^2b^2=2\left(\lambda^2-1\right)x+1-\lambda^2 \end{cases} \Rightarrow a^2-\lambda^2b^2=a-\lambda b$

$\Leftrightarrow \left(a-\lambda b\right)\left(a+\lambda b-1\right)=0$.

Cazul1).Dacă $a=\lambda b \Rightarrow \sqrt{\lambda^2x^2+\left(\lambda^2-2\right)x+1}=\lambda\sqrt{x^2-x+1} \Rightarrow x=\frac{1}{2}$, care convine.

Cazul2).Dacă $a=1-\lambda b \Rightarrow \sqrt{\lambda^2x^2+\left(\lambda^2-2\right)x+1}=1-\lambda\sqrt{x^2-x+1}$, care nu convine,

deoarece $1-\lambda\sqrt{x^2-x+1}<0 \Leftrightarrow \lambda\sqrt{x^2-x+1}>1 \Leftrightarrow \lambda^2x^2-\lambda^2x+\lambda^2-1>0$,

vezi $\Delta=\lambda^2\left(4-3\lambda^2\right)<0, \lambda>\frac{2}{\sqrt{3}}$.

Deducem că ecuația are singura soluție reală $x=\frac{1}{2}$.

Problema105.

If $a,b,c>0$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$ then

$$\left(a+b+c\right)\left(ab+bc+ca\right)+abc+\frac{1}{abc}\geq 11.$$

Soluție.

Folosim pqr -Method.

Notăm $p=a+b+c, q=ab+bc+ca, r=abc$.

Avem: $3=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 3\sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}}=\frac{3}{\sqrt[3]{r}} \Rightarrow r\geq 1$.

Avem $p\geq 3\sqrt[3]{r}$, $q\geq 3\sqrt[3]{r^2} \Rightarrow pq\geq 9r$.

Inegalitatea se scrie $pq+r+\frac{1}{r}\geq 11$, care rezultă din $pq\geq 9r$ și $r\geq 1 \Rightarrow pq\geq 9$.

Obținem $pq+\left(r+\frac{1}{r}\right)\overset{AM-GM}{\geq} 9+2=11$.

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Remarca.

If $a,b,c>0$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$ and $0\leq\lambda\leq 10$ then

$$\left(a+b+c\right)\left(ab+bc+ca\right)+abc+\frac{\lambda}{abc}\geq \lambda+10.$$

Marin Chirciu

Soluție.

Folosim pqr -Method.

Notăm $p=a+b+c, q=ab+bc+ca, r=abc$.

Avem: $3=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 3\sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}}=\frac{3}{\sqrt[3]{r}}\Rightarrow r\geq 1$.

Avem $p\geq 3\sqrt[3]{r}$, $q\geq 3\sqrt[3]{r^2}\Rightarrow pq\geq 9r$.

Inegalitatea se scrie $pq+r+\frac{1}{r}\geq 11$, care rezultă din $pq\geq 9r$ și $r\geq 1$.

Obținem $pq+r+\frac{\lambda}{r}\geq 9r+r+\frac{\lambda}{r}\overset{(1)}{\geq}\lambda+10$, unde $(1)\Leftrightarrow 10r+\frac{\lambda}{r}\geq\lambda+10\Leftrightarrow$

$\Leftrightarrow 10r^2-(\lambda+10)r+\lambda\geq 0\Leftrightarrow (r-1)(10r-\lambda)\geq 0$, care rezultă din $r\geq 1\geq\frac{\lambda}{10}$,

vezi $0\leq\lambda\leq 10$.

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Remarca.

If $a,b,c>0$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$ then

$$(a+b+c)(ab+bc+ca)+abc+\frac{10}{abc}\geq 20.$$

Problema106.

In acute ΔABC

$$\left(\sum\tan A\right)^2\geq\sum(1+\sec A)^2.$$

Mathematical Inequalities5/2024,George Apostolopoulos, Greece

Soluție.

Cu substituția $A\to\frac{\pi}{2}-\frac{A}{2}$, problema se reformulează.

In ΔABC

$$\left(\sum\cot\frac{A}{2}\right)^2\geq\sum\left(1+\csc\frac{A}{2}\right)^2.$$

Demonstrație.

$$\left(\sum\cot\frac{A}{2}\right)^2\geq\sum\left(1+\csc\frac{A}{2}\right)^2\Leftrightarrow\left(\frac{p}{r}\right)^2\geq\sum\left(1+\frac{1}{\sin\frac{A}{2}}\right)^2\Leftrightarrow\frac{p^2}{r^2}\geq\sum\left(1+\frac{2}{\sin\frac{A}{2}}+\frac{1}{\sin^2\frac{A}{2}}\right)$$

$$\Leftrightarrow\frac{p^2}{r^2}\geq 3+\sum\frac{2}{\sin\frac{A}{2}}+\sum\frac{1}{\sin^2\frac{A}{2}}\Leftrightarrow\frac{p^2}{r^2}\geq 3+\sum\frac{2}{\sin\frac{A}{2}}+\frac{p^2+r^2-8Rr}{r^2}\Leftrightarrow$$

$\Leftrightarrow\sum\frac{1}{\sin\frac{A}{2}}\leq 2\left(\frac{2R}{r}-1\right)$, care rezultă din inegalitatea CBS:

$$\sum\frac{1}{\sin\frac{A}{2}}\overset{CBS}{\leq}\sqrt{3\sum\frac{1}{\sin^2\frac{A}{2}}}=\sqrt{3\cdot\frac{p^2+r^2-8Rr}{r^2}}\overset{(1)}{\leq}2\left(\frac{2R}{r}-1\right),$$

unde (1) $\Leftrightarrow \sqrt{3\cdot\frac{p^2+r^2-8Rr}{r^2}} \le 2\left(\frac{2R}{r}-1\right) \Leftrightarrow 3p^2 \le 16R^2+8Rr+r^2$, vezi inegalitatea lui Gerretsen $p^2 \le 4R^2+4Rr+3r^2$.

Rămâne să arătăm că:

$3\left(4R^2+4Rr+3r^2\right) \le 16R^2+8Rr+r^2 \Leftrightarrow R^2-Rr-2r^2 \ge 0 \Leftrightarrow (R-2r)(R+r) \ge 0$,

vezi $R \ge 2r$,(Euler).

Am folosit mai sus $\sum \cot\frac{A}{2} = \frac{p}{r}$ și $\sum \frac{1}{\sin^2\frac{A}{2}} = \frac{p^2+r^2-8Rr}{r^2}$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema107.

If $a,b,c>0$, $a+b+c=3$ then

$$\sum \frac{a}{b^2+c^2+2a+2} \le \frac{1}{2}.$$

RMM 5/2024, Nguyen Hung Cuong, Vietnam

Soluție.

Lema.

If $a,b,c>0$ then

$$\frac{a}{b^2+c^2+2a+2} \le \frac{a}{2(a+b+c)}.$$

Demonstrație.

$\frac{a}{b^2+c^2+2a+2} = \frac{a}{(b^2+1)+(c^2+1)+2a} \overset{SOS}{\le} \frac{a}{2b+2c+2a} = \frac{a}{2(a+b+c)}$, cu egalitate pentru $b=c=1$.

$LHS = \sum \frac{a}{b^2+c^2+2a+2} \overset{Lema}{\le} \sum \frac{a}{2(a+b+c)} = \frac{1}{2} = RHS$,

Egalitatea are loc dacă și numai dacă $a=b=c=1$.

Remarca.

If $a,b,c,d>0$, $a+b+c+d=4$ then

$$\sum \frac{a}{b^2+c^2+d^2+2a+3} \le \frac{1}{2}.$$

Marin Chirciu

Soluție.

Lema.

If $a,b,c>0$ then

$$\frac{a}{b^2+c^2+d^2+2a+3} \le \frac{a}{2(a+b+c)}.$$

Demonstrație.

$$\frac{a}{b^2+c^2+d^2+2a+3} = \frac{a}{(b^2+1)+(c^2+1)+(d^2+1)+2a} \overset{SOS}{\le} \frac{a}{2b+2c++2d+2a} = \frac{a}{2(a+b+c+d)}$$

cu egalitate pentru $b=c=d=1$.

$LHS = \sum \frac{a}{b^2+c^2+d^2+2a+3} \overset{Lema}{\le} \sum \frac{a}{2(a+b+c+d)} = \frac{1}{2} = RHS$,

Egalitatea are loc dacă și numai dacă $a=b=c=d=1$.

Problema108.

If $a,b>0$ then

$$4\left(\frac{a^2}{b+1}+\frac{b^2}{a+1}\right)+3\left(\frac{1}{a}+\frac{1}{b}\right)\geq 10.$$

Nguyen Hung Cuong, Vietnam

Soluție.

$$LHS=4\left(\frac{a^2}{b+1}+\frac{b^2}{a+1}\right)+3\left(\frac{1}{a}+\frac{1}{b}\right)\overset{CS}{\geq}4\frac{(a+b)^2}{a+b+2}+3\frac{(1+1)^2}{a+b}\overset{t=a+b}{=}\frac{4t^2}{t+2}+\frac{12}{t}\overset{(1)}{\geq}10=RHS,$$

unde $(1)\Leftrightarrow \frac{4t^2}{t+2}+\frac{12}{t}\geq 10 \Leftrightarrow 2t^3-5t^2-4t+12\geq 0\Leftrightarrow (t-2)^2(2t+3)\geq 0$.

Egalitatea are loc dacă și numai dacă $a=b=1$.

Remarca.

If $a,b>0$ then

$$1).\ 4\left(\frac{a^3}{b+1}+\frac{b^3}{a+1}\right)+5\left(\frac{1}{a}+\frac{1}{b}\right)\geq 14.$$

$$2).\ 4\left(\frac{a^4}{b+1}+\frac{b^4}{a+1}\right)+7\left(\frac{1}{a}+\frac{1}{b}\right)\geq 18.$$

$$3).\ 4\left(\frac{a^5}{b+1}+\frac{b^5}{a+1}\right)+9\left(\frac{1}{a}+\frac{1}{b}\right)\geq 22.$$

Remarca.

If $a,b>0$ and $n\in\mathbf{N}, n\geq 2$ then

$$4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)+(2n-1)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4n+2.$$

Marin Chirciu

Soluție.

$$LHS=4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)+(2n-1)\left(\frac{1}{a}+\frac{1}{b}\right)\overset{Holder}{\geq}4\frac{(a+b)^n}{2^{n-2}(a+b+2)}+(2n-1)\frac{(1+1)^2}{a+b}\overset{t=a+b}{=}$$

$$\overset{t=a+b}{=}\frac{t^n}{2^{n-4}(t+2)}+\frac{4(2n-1)}{t}\overset{(1)}{\geq}4n+2=RHS,$$

unde $(1)\Leftrightarrow \frac{t^n}{2^{n-4}(t+2)}+\frac{4(2n-1)}{t}\geq 4n+2$

$$\Leftrightarrow 8t^{n+1}-2^n(2n+1)t^2-4\cdot 2^n t+2^n(8n-4)\geq 0\Leftrightarrow$$

$$\Leftrightarrow (t-2)^2\left(8t^{n-1}+32t^{n-2}+96t^{n-3}+\ldots+(n-2)2^n t^2+(n-1)2^{n+1}t+2^n\left(2^n-1\right)\right)\geq 0.$$

Egalitatea are loc dacă și numai dacă $a=b=1$.

Problema109.

If $x,y,z>0$, $xy+yz+zx=3$ then

$$\sum\frac{1}{x^2+2}\leq 1.$$

Mathematical Inequalities 5/2024, Nguyen Viet Hung, Vietnam

Soluție.

$\sum \frac{1}{x^2+2} \le 1 \Leftrightarrow \sum \frac{x^2}{x^2+2} \ge 1$, care rezultă din inegalitatea lui Bergstrom:

$$\sum \frac{x^2}{x^2+2} \overset{CS}{\ge} \frac{\left(\sum x\right)^2}{\sum\left(x^2+2\right)} = \frac{\left(\sum x\right)^2}{\sum x^2+6} = \frac{\left(\sum x\right)^2}{\sum x^2 + 2\cdot\sum yz} = \frac{\left(\sum x\right)^2}{\left(\sum x\right)^2} = 1.$$

Egalitatea are loc dacă și numai dacă $x = y = z = 1$.

Remarca.

In ΔABC

$$\sum \frac{1}{3\tan^2\frac{A}{2}+2} \le 1.$$

Marin Chirciu

Soluție.

Lema.

If $x, y, z > 0$, $xy + yz + zx = 3$ then

$$\sum \frac{1}{x^2+2} \le 1.$$

Se cunoaște identitatea în triunghi $\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1 \Leftrightarrow \sum \sqrt{3}\tan\frac{B}{2}\cdot\sqrt{3}\tan\frac{C}{2} = 3$.

Folosind **Lema** pentru $(x, y, z) = \left(\sqrt{3}\tan\frac{A}{2}, \sqrt{3}\tan\frac{B}{2}, \sqrt{3}\tan\frac{C}{2}\right)$ obținem:

$$\sum \frac{1}{3\tan^2\frac{A}{2}+2} \le 1.$$

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Remarca.

If $x, y, z > 0$, $xy + yz + zx = 3$ and $\lambda \ge 2$ then

$$\sum \frac{1}{x^2+\lambda} \le \frac{3}{\lambda+1}.$$

Marin Chirciu

Soluție.

$\sum \frac{1}{x^2+\lambda} \le \frac{3}{\lambda+1} \Leftrightarrow \sum \frac{1}{x^2+\lambda} \ge \frac{3}{\lambda+1}$, care rezultă din inegalitatea lui Bergstrom:

$$\sum \frac{x^2}{x^2+\lambda} \overset{CS}{\ge} \frac{\left(\sum x\right)^2}{\sum\left(x^2+\lambda\right)} = \frac{\left(\sum x\right)^2}{\sum x^2+3\lambda} = \frac{\sum x^2 + 2\sum yz}{\sum x^2 + \lambda\cdot\sum yz} \overset{(1)}{\ge} \frac{3}{\lambda+1},$$

unde(1) $\Leftrightarrow \frac{\sum x^2 + 2\sum yz}{\sum x^2 + \lambda\sum yz} \ge \frac{3}{\lambda+1} \Leftrightarrow (\lambda-2)\sum x^2 \ge (\lambda-2)\sum yz$, vezi $\lambda \ge 2$ și

$\sum x^2 \ge \sum yz \Leftrightarrow \sum (x-y)^2 \ge 0$.

Egalitatea are loc dacă și numai dacă $x = y = z = 1$.

Problema110.

S:L24.143. Calculați

$$\lim_{n\to\infty}\left(\frac{1^3}{n^4+1^3} + \frac{2^3}{n^4+2^3} + \ldots + \frac{n^3}{n^4+n^3}\right).$$

SGM4-2024, Gheorghe Crăciun, Ploiești

Soluție.

Folosim Teorema”Cleștelui”pentru șirul $a_n=\frac{1^3}{n^4+1^3}+\frac{2^3}{n^4+2^3}+...+\frac{n^3}{n^4+n^3}$.

Avem $\sum_{k=1}^{n}\frac{k^3}{n^4+n^3}<\sum_{k=1}^{n}\frac{k^3}{n^4+k^3}<\sum_{k=1}^{n}\frac{k^3}{n^4+1^3}$.

Obținem $\sum_{k=1}^{n}\frac{k^3}{n^4+1^3}=\frac{\sum_{k=1}^{n}k^3}{n^4+1}=\frac{\frac{n^2(n+1)^2}{4}}{n^4+1}\to\frac{1}{4}$ și $\sum_{k=1}^{n}\frac{k^3}{n^4+n^3}=\frac{\sum_{k=1}^{n}k^3}{n^4+1}=\frac{\frac{n^2(n+1)^2}{4}}{n^4+n^3}\to\frac{1}{4}$.

Rezultă $\lim_{n\to\infty}\left(\frac{1^3}{n^4+1^3}+\frac{2^3}{n^4+2^3}+...+\frac{n^3}{n^4+n^3}\right)=\frac{1}{4}$.

Remarca.

Calculați

$$\lim_{n\to\infty}\left(\frac{1^3}{\lambda n^4+1^3}+\frac{2^3}{\lambda n^4+2^3}+...+\frac{n^3}{\lambda n^4+n^3}\right),\text{unde }\lambda>0.$$

Marin Chirciu

Soluție.

Folosim Teorema”Cleștelui”pentru șirul $a_n=\frac{1^3}{\lambda n^4+1^3}+\frac{2^3}{\lambda n^4+2^3}+...+\frac{n^3}{\lambda n^4+n^3}$.

Avem $\sum_{k=1}^{n}\frac{k^3}{\lambda n^4+n^3}<\sum_{k=1}^{n}\frac{k^3}{\lambda n^4+k^3}<\sum_{k=1}^{n}\frac{k^3}{\lambda n^4+1^3}$.

Obținem:

$$\sum_{k=1}^{n}\frac{k^3}{\lambda n^4+1^3}=\frac{\sum_{k=1}^{n}k^3}{\lambda n^4+1}=\frac{\frac{n^2(n+1)^2}{4}}{\lambda n^4+1}\to\frac{1}{4\lambda}\text{ și }\sum_{k=1}^{n}\frac{k^3}{\lambda n^4+n^3}=\frac{\sum_{k=1}^{n}k^3}{\lambda n^4+1}=\frac{\frac{n^2(n+1)^2}{4}}{\lambda n^4+n^3}\to\frac{1}{4\lambda}.$$

Rezultă $\lim_{n\to\infty}\left(\frac{1^3}{\lambda n^4+1^3}+\frac{2^3}{\lambda n^4+2^3}+...+\frac{n^3}{\lambda n^4+n^3}\right)=\frac{1}{4\lambda}$.

Problema111.

S662. In ΔABC

$$\text{a) }\frac{1}{r}+\frac{9}{4R+r}\geq 4\sum\frac{1}{r_a+r_b}.$$

$$\text{b) }2R+5r\geq h_a+h_b+h_c$$

Mathematical Reflections 3/2024, Nguyen Viet Hung, Vietnam

Solution.

a)Using Popoviciu `Inequality for convex function $f(x)=\frac{1}{x}, x>0$ we have:

$$\frac{f(x)+f(y)+f(z)}{3}+f\left(\frac{x+y+z}{3}\right)\geq\frac{2}{3}\sum f\left(\frac{x+y}{2}\right)\Leftrightarrow$$

$$\Leftrightarrow\frac{1}{3}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+\frac{3}{x+y+z}\geq\frac{2}{3}\sum\frac{2}{x+y}.$$

Putting $(x,y,z)=(r_a,r_b,r_c)$ and $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}, r_a+r_b+r_c=4R+r$ we have:

$$\frac{1}{3}\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)+\frac{3}{r_a+r_b+r_c}\geq\frac{2}{3}\sum\frac{2}{r_a+r_b}\Leftrightarrow\frac{1}{3}\cdot\frac{1}{r}+\frac{3}{4R+r}\geq\frac{2}{3}\sum\frac{2}{r_a+r_b}\Leftrightarrow$$

$\Leftrightarrow \frac{1}{r}+\frac{9}{4R+r}\geq 4\sum \frac{1}{r_a+r_b}$.

Equality occurs if and only if triangle is equilateral.

b) $h_a+h_b+h_c=\frac{p^2+r^2+4Rr}{2R}\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2+r^2+4Rr}{2R}=\frac{2R^2+4Rr+2r^2}{R}\overset{(1)}{\leq}2R+5r$,

(1) $\Leftrightarrow \frac{2R^2+4Rr+2r^2}{R}\leq 2R+5r \Leftrightarrow R\geq 2r$,(Euler).

Equality occurs if and only if triangle is equilateral.

Problema112.

If $a,b,c,d,e>0$ then

$$2\sum \frac{a^3+1}{a^2+1}+\frac{1}{abcde}\geq 11.$$

RMM 5/2024, Kunihiko Chikaya

Soluție.

Lema.

If $a>0$ then

$$\frac{a^3+1}{a^2+1}\geq \frac{a+1}{2}.$$

Demonstrație.

$\frac{a^3+1}{a^2+1}\geq \frac{a+1}{2} \Leftrightarrow (a-1)^2(a+1)\geq 0$, cu egalitate pentru $a=1$.

$LHS=2\sum \frac{a^3+1}{a^2+1}+\frac{1}{abcde}\overset{Lema}{\geq}\sum (a+1)+\frac{1}{abcde}=5+\sum a+\frac{1}{abcde}\overset{AM-GM}{\geq}5+6\sqrt[6]{abcde\cdot \frac{1}{abcde}}=$

$=5+6=11=RHS$

Egalitatea are loc daca si numai daca $a=b=c=d=e=1$.

Remarca.

If $a,b,c,d>0$ and $\lambda \geq 0$ then

$$(\lambda+1)\sum \frac{a^3+\lambda}{a^2+\lambda}+\frac{1}{abcd}\geq 4\lambda+5.$$

Remarca.

If $a,b,c>0$ and $\lambda \geq 0$ then

$$(\lambda+1)\sum \frac{a^3+\lambda}{a^2+\lambda}+\frac{1}{abc}\geq 3\lambda+4.$$

Remarca.

If $a_1,a_2,...,a_n>0$ and $\lambda \geq 0$ then

$$(\lambda+1)\sum \frac{a_1^3+\lambda}{a_1^2+\lambda}+\frac{1}{a_1a_2...a_n}\geq n\lambda+n+1.$$

Marin Chirciu

Soluție.

Lema.

If $a>0$ then

$$\frac{a^3+\lambda}{a^2+\lambda}\geq \frac{a+\lambda}{\lambda+1}.$$

$$(\lambda+1)\sum\frac{a_1^3+\lambda}{a_1^2+\lambda}+\frac{1}{a_1a_2...a_n}\overset{Lema}{\geq}\sum(a_1+\lambda)+\frac{1}{a_1a_2...a_n}=n\lambda+\sum a_1+\frac{1}{a_1a_2...a_n}\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq} n\lambda+(n+1)\sqrt[n+1]{a_1a_2...a_n\cdot\frac{1}{a_1a_2...a_n}}=n\lambda+n+1=RHS$$

Egalitatea are loc daca si numai daca $a_1=a_2=...=a_n=1$.

Problema113.

If $a,b,c>0$, $abc=1$ then

$$\sum\frac{1}{a^2+2}\leq 1.$$

Soluție.

$\sum\frac{1}{a^2+2}\leq 1 \Leftrightarrow \sum(b^2+2)(c^2+2)\leq\prod(a^2+2) \Leftrightarrow a^2b^2c^2+\sum b^2c^2\geq 4$, vezi AM-GM.

Obținem $a^2b^2c^2+a^2b^2+b^2c^2+c^2a^2\geq 4\sqrt[4]{a^6b^6c^6}=4$.

Egalitatea are loc daca si numai daca $a=b=c=1$.

Remarca.

If $a,b,c>0$, $abc=1$ and $\lambda\geq 2$ then

$$\sum\frac{1}{a^2+\lambda}\leq\frac{3}{\lambda+1}.$$

Marin Chirciu

Soluție.

$\sum\frac{1}{a^2+\lambda}\leq\frac{3}{\lambda+1} \Leftrightarrow (\lambda+1)\sum(b^2+2)(c^2+2)\leq 3\prod(a^2+2) \Leftrightarrow$

$3a^2b^2c^2+(2\lambda-1)\sum b^2c^2+(\lambda^2-2\lambda)\sum a^2\geq 3\lambda^2$, care rezultă din AM-GM și $abc=1$.

Obținem:

$$3a^2b^2c^2+(2\lambda-1)\sum b^2c^2+(\lambda^2-2\lambda)\sum a^2\overset{AM-GM}{\geq}3a^2b^2c^2+(2\lambda-1)\cdot 3\sqrt[3]{a^4b^4c^4}+$$

$$+(\lambda^2-2\lambda)\cdot 3\sqrt[3]{a^2b^2c^2}\overset{(1)}{\geq}3\lambda^2,$$

unde (1) $\Leftrightarrow a^2b^2c^2+(2\lambda-1)\cdot\sqrt[3]{a^4b^4c^4}+(\lambda^2-2\lambda)\cdot\sqrt[3]{a^2b^2c^2}\geq\lambda^2$, care rezultă din:

Notând $\sqrt[3]{a^2b^2c^2}=t \Leftrightarrow a^2b^2c^2=t^3$ rezultă:

$t^3+(2\lambda-1)t^2+(\lambda^2-2\lambda)t-\lambda^2\geq 0 \Leftrightarrow (t-1)(t+\lambda)^2\geq 0$, vezi $t\geq 1 \Leftrightarrow \sqrt[3]{a^2b^2c^2}\geq 1$.

Egalitatea are loc daca si numai daca $a=b=c=1$.

Problema114.

In ΔABC

$$\frac{18r}{R}\leq\sum a\sum\frac{1}{a}\leq\left(\frac{3R}{2r}\right)^2.$$

RMM5/2020, Eldeniz Hesenov, Georgia

Soluție.

Lema.

In ΔABC

$$\sum a\sum\frac{1}{a}=\frac{p^2+r^2+4Rr}{2Rr}.$$

Inegalitatea din dreapta.

$$\sum a\sum\frac{1}{a}=\frac{p^2+r^2+4Rr}{2Rr}\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2+r^2+4Rr}{2Rr}=\frac{4R^2+8Rr+4r^2}{2Rr}=$$
$$=\frac{2\left(R^2+2Rr+r^2\right)}{Rr}\overset{Euler}{\leq}\frac{9R}{2r}\overset{Euler}{\leq}\left(\frac{3R}{2r}\right)^2.$$

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Inegalitatea din stânga.

$$\sum a\sum\frac{1}{a}=\frac{p^2+r^2+4Rr}{2Rr}\overset{Gerretsen}{\geq}\frac{16Rr-5r^2+r^2+4Rr}{2Rr}=\frac{20Rr-4r^2}{2Rr}=2\left(5-\frac{r}{R}\right)\overset{Euler}{\geq}\frac{18r}{R}.$$

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Remarca.

In ΔABC

$$10-\frac{2r}{R}\leq\sum a\sum\frac{1}{a}\leq\frac{9R}{2r}.$$

Marin Chirciu

Remarca.

In ΔABC

$$\frac{18r}{R}\leq 10-\frac{2r}{R}\leq\sum a\sum\frac{1}{a}\leq\frac{9R}{2r}\leq\left(\frac{3R}{2r}\right)^2.$$

Problema115.

If $a,b,c>0$, $a+b+c+2=abc$ then

$$\sum\frac{a^2}{a+1}\geq 4.$$

Mathematical Inequalities5/2024, Nguyen Minh Tho, Vietnam

Soluție.

Lema.

If $a,b,c>0$, $a+b+c+2=abc$ then

$$a+b+c\geq 6.$$

Demonstrație.

$$p=a+b+c=abc-2\overset{AM-GM}{\leq}\left(\frac{a+b+c}{3}\right)^3-2=\left(\frac{p}{3}\right)^3-2\Rightarrow p\geq\left(\frac{p}{3}\right)^3-2\Leftrightarrow$$
$\Leftrightarrow p^3-27p-54\geq 0\Leftrightarrow (p-6)(p+3)^2\geq 0\Leftrightarrow p\geq 6$, cu egalitate pentru $a=b=c=2$.

$LHS=\sum\frac{a^2}{a+1}\overset{CS}{\geq}\frac{\left(\sum a\right)^2}{\sum(a+1)}=\frac{\left(\sum a\right)^2}{\sum a+3}\overset{(1)}{\geq}4=RHS$, unde (1) $\Leftrightarrow\frac{\left(\sum a\right)^2}{\sum a+3}\geq 4\Leftrightarrow\frac{p^2}{p+3}\geq 4\Leftrightarrow$

$\Leftrightarrow p^2-4p-12\geq 0\Leftrightarrow (p-6)(p+2)\geq 0$, vezi $p\geq 6$.

Egalitatea are loc daca si numai daca $a=b=c=2$.

Remarca.

If $a,b,c>0$, $a+b+c+2=abc$ and $\lambda\geq 0$ then

$$1).\ \sum\frac{a^2}{a+\lambda}\geq\frac{12}{\lambda+2}.$$
$$2).\ \sum\frac{a^3}{a+\lambda}\geq\frac{24}{\lambda+2}.$$
$$3).\ \sum\frac{a^4}{a+\lambda}\geq\frac{24}{\lambda+2}.$$

Remarca.

If $a,b,c>0$, $a+b+c+2=abc$ and $\lambda\ge 0$, $n\in\mathbf{N}, n\ge 2$ then

$$\sum\frac{a^n}{a+\lambda}\ge\frac{3\cdot 2^n}{\lambda+2}.$$

Marin Chirciu

Soluție.

Lema.

If $a,b,c>0$, $a+b+c+2=abc$ then

$$a+b+c\ge 6.$$

$$LHS=\sum\frac{a^n}{a+\lambda}\overset{CS}{\ge}\frac{\left(\sum a\right)^n}{3^{n-2}\sum(a+\lambda)}=\frac{\left(\sum a\right)^n}{3^{n-2}\left(\sum a+3\lambda\right)}\overset{(1)}{\ge}\frac{3\cdot 2^n}{\lambda+2}=RHS \text{ , unde (1)}$$

$$\Leftrightarrow\frac{\left(\sum a\right)^n}{3^{n-2}\left(\sum a+3\lambda\right)}\ge\frac{3\cdot 2^n}{\lambda+2}\Leftrightarrow\frac{p^n}{3^{n-2}(p+3\lambda)}\ge\frac{3\cdot 2^n}{\lambda+2}\Leftrightarrow$$

$$\Leftrightarrow(\lambda+2)p^n-2^n\cdot 3^{n-1}p-6^n\lambda\ge 0\Leftrightarrow$$

$$\Leftrightarrow(p-6)\left[(\lambda+2)p^{n-1}+2(\lambda+2)p^{n-2}+\ldots+2(\lambda+2)6^{n-2}p+6^{n-1}\lambda\right]\ge 0 \text{ , vezi } p\ge 6.$$

Egalitatea are loc daca si numai daca $a=b=c=2$.

Problema116.

S:E24.75 Rezolvați în numere întregi ecuația

$$x^2-3xy+y+1=0.$$

SGM-2/2024, Cristian Amorăriței, Suceava

Soluție.

$$x^2-3xy+y+1=0\Leftrightarrow y=\frac{2x^2+1}{3x-1}\in\mathbf{Z}\Leftrightarrow y=\frac{1}{9}\left(6x+2+\frac{11}{3x-1}\right)\Rightarrow\frac{11}{3x-1}\in\mathbf{Z}\Leftrightarrow$$

$$\Leftrightarrow 3x-1\in M_{11}\Leftrightarrow 3x-1\in\{-11,-1,1,11\}\Rightarrow x\in\{0,4\}\Rightarrow y\in\{-1,3\}.$$

Mulțimea soluțiilor ecuației este $S=\{(0,-1),(4,3)\}$.

Problema117.

S:E24.80 Rezolvați ecuația

$$2x^5-x^4+4x^3-2x^2+2x=2024.$$

SGM-2/2024, Alexandru Moscaliuc, Botoșani

Soluție.

$$2x^5-x^4+4x^3-2x^2+2x=2024\Leftrightarrow(x-4)\left(2x^4+7x^3+32x^2+126x+506\right)=0.$$

Ecuația admite singura soluție reală $x=4$, deoarece $2x^4+7x^3+32x^2+126x+506>0$, vezi

$$2x^4+7x^3+32x^2+126x+506=x^2\underbrace{\left(2x^2+7x+7\right)}_{>0,\Delta<0}+\underbrace{\left(25x^2+126x+506\right)}_{>0,\Delta<0}>0.$$

Problema118.

If $a,b>0$ then

$$\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{a}}\le\sqrt[3]{2(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}.$$

Mathematics(College and High School) 5/2024, Amir Sofi, Kosovo

Soluție.

Cu substituția $(a,b)=\left(x^3,y^3\right)$ inegalitatea se scrie:

$\frac{x}{y}+\frac{y}{x}\le\sqrt[3]{2\left(x^3+y^3\right)\left(\frac{1}{x^3}+\frac{1}{y^3}\right)}\Leftrightarrow x^2+y^2\le\sqrt[3]{2\left(x^3+y^3\right)^2}$, care rezultă din inegalitatea ponderată a mediilor $\sqrt{\frac{x^2+y^2}{2}}\le\sqrt[3]{\frac{x^3+y^3}{2}}$.

Egalitatea are loc daca si numai daca $a=b$.

Remarca.

If $a,b>0$ then

$$1).\ \sqrt[4]{\frac{a}{b}}+\sqrt[4]{\frac{b}{a}}\le\sqrt[4]{4(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}.$$

$$2).\ \sqrt[5]{\frac{a}{b}}+\sqrt[5]{\frac{b}{a}}\le\sqrt[5]{8(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}.$$

Remarca.

If $a,b>0$ and $n\in\mathbf{N},n\ge2$ then

$$\sqrt[n]{\frac{a}{b}}+\sqrt[n]{\frac{b}{a}}\le\sqrt[n]{2^{n-2}(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}.$$

Marin Chirciu

Soluție.

Cu substituția $(a,b)=\left(x^n,y^n\right)$ inegalitatea se scrie:

$\frac{x}{y}+\frac{y}{x}\le\sqrt[n]{2^{n-2}\left(x^n+y^n\right)\left(\frac{1}{x^n}+\frac{1}{y^n}\right)}\Leftrightarrow x^2+y^2\le\sqrt[n]{2^{n-2}\left(x^n+y^n\right)^2}$,

vezi $\sqrt{\frac{x^2+y^2}{2}}\le\sqrt[5]{\frac{x^5+y^5}{2}}$,vezi inegalitatea ponderată $\sqrt{\frac{x^2+y^2}{2}}\le\sqrt[n]{\frac{x^n+y^n}{2}}$, $n\in\mathbf{N},n\ge2$.

Egalitatea are loc daca si numai daca $a=b$ sau $n=2$.

Problema119.

If $a,b,c>0$, $abc=1$ then

$$(a+b+c)^2\left(\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\right)\ge9.$$

RMM 5/2024, Nguyen Hung Cuong, Vietnam

Soluție.

Avem $(a+b+c)^2=\sum a^2+2\sum bc\overset{AM-GM}{\ge}\sum a^2+2\cdot3\sqrt[3]{a^2b^2c^2}\overset{abc=1}{=}\sum a^2+6=\sum\left(a^2+2\right)$.

Obținem $LHS=(a+b+c)^2\left(\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\right)\ge\sum\left(a^2+2\right)\sum\frac{1}{a^2+2}\overset{CS}{\ge}9=RHS$.

Egalitatea are loc daca si numai daca $a=b=c=1$.

Remarca.

If $a,b,c>0$, $abc=1$ then

$$1).\ (a+b+c)^3\left(\frac{1}{a^3+8}+\frac{1}{b^3+8}+\frac{1}{c^3+8}\right)\ge9.$$

$$2).\ (a+b+c)^4\left(\frac{1}{a^4+26}+\frac{1}{b^4+26}+\frac{1}{c^4+26}\right)\ge9.$$

$$3).\ (a+b+c)^5\left(\frac{1}{a^5+26}+\frac{1}{b^5+26}+\frac{1}{c^5+26}\right)\ge9.$$

Dezvoltări, Marin Chirciu

Soluție.

Avem $(a+b+c)^5 = a^5+b^5+c^5+5(a+b)(b+c)(c+a)(a^2+b^2+c^2+ab+bc+ca) \overset{AM-GM}{\geq}$

$\overset{AM-GM}{\geq} \sum a^5 + 5\cdot\prod(2\sqrt{bc})\cdot\left(3\sqrt[3]{a^2b^2c^2}+3\sqrt[3]{a^2b^2c^2}\right) \overset{abc=1}{=} \sum a^5 + 5\cdot 8\cdot 6 = \sum a^5 + 240 =$

$= \sum(a^5+80)$.

Obținem $LHS = (a+b+c)^5\left(\frac{1}{a^5+80}+\frac{1}{b^5+80}+\frac{1}{c^5+80}\right) \geq \sum(a^5+80)\sum\frac{1}{a^5+80} \overset{CS}{\geq} 9 = RHS$.

Egalitatea are loc daca si numai daca $a=b=c=1$.

Problema120.

If $a,b,c>0$, $\sum\sqrt[3]{a^3+7}=6$ then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 3.$$

Mathematical Inequalities 5/2024, Kunihiko Chikaya

Soluție.

Lema.

If $a>0$, then

$$\sqrt[3]{a^3+7}\geq\frac{a+7}{4}.$$

Demonstrație.

$\sqrt[3]{a^3+7}\geq\frac{x+7}{4} \Leftrightarrow 64(a^3+7)\geq(a+7)^3 \Leftrightarrow 3a^3-a^2-7a+5\geq 0 \Leftrightarrow (a-1)^2(3a+5)\geq 0$.

$6=\sum\sqrt[3]{a^3+7} \overset{Lema}{\geq} \sum\frac{a+7}{4} = \frac{\sum a+21}{4} = \frac{1}{4}\sum a+\frac{21}{4} \overset{CS}{\geq} \frac{1}{4}\cdot\frac{9}{\sum\frac{1}{a}}+\frac{21}{4} \Rightarrow \sum\frac{1}{a}\geq 3.$

Egalitatea are loc daca si numai daca $a=b=c=1$.

Remarca.

If $a,b,c>0$, $\sum\sqrt[3]{a^3+\lambda}=3\sqrt[3]{\lambda+1}$, $\lambda\geq 0$ then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 3.$$

Marin Chirciu

Soluție.

Lema.

If $a>0$ and $\lambda\geq 0$ then

$$\sqrt[3]{a^3+\lambda}\geq\frac{a+\lambda}{\sqrt[3]{(\lambda+1)^2}}.$$

Demonstrație.

$\sqrt[3]{a^3+\lambda}\geq\frac{a+\lambda}{\sqrt[3]{(\lambda+1)^2}} \Leftrightarrow (\lambda+1)^2(a^3+\lambda)\geq(a+\lambda)^3 \Leftrightarrow (\lambda+2)a^3-3a^2-3\lambda a+2\lambda+1\geq 0 \Leftrightarrow$

$(a-1)^2[(\lambda+2)a+2\lambda+1]\geq 0$, cu egalitate pentru $a=1$.

$$3\sqrt[3]{\lambda+1}=\sqrt[3]{a^3+\lambda}\overset{Lema}{\geq}\sum\frac{a+\lambda}{\sqrt[3]{(\lambda+1)^2}}=\frac{\sum a+3\lambda}{\sqrt[3]{(\lambda+1)^2}}=\frac{1}{\sqrt[3]{(\lambda+1)^2}}\sum a+\frac{3\lambda}{\sqrt[3]{(\lambda+1)^2}}\overset{CS}{\geq}$$

$$\overset{CS}{\geq}\frac{1}{\sqrt[3]{(\lambda+1)^2}}\cdot\frac{9}{\sum\frac{1}{a}}+\frac{3\lambda}{\sqrt[3]{(\lambda+1)^2}}\Rightarrow\sum\frac{1}{a}\geq 3.$$

Egalitatea are loc daca si numai daca $a=b=c=1$.

Remarca.

If $a,b,c>0$, $\sum\sqrt[3]{a^3+26}=9$ then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 3.$$

Marin Chirciu

Soluție.

Lema.

If $a>0$ then

$$\sqrt[3]{a^3+26}\geq\frac{a+26}{9}.$$

Demonstrație.

$$\sqrt[3]{a^3+26}\geq\frac{a+26}{9}\Leftrightarrow 729\left(a^3+26\right)\geq(a+26)^3\Leftrightarrow 28a^3-3a^2-78a+53\geq 0\Leftrightarrow$$

$(a-1)^2(28a+53)\geq 0$, cu egalitate pentru $a=1$.

$$9=\sum\sqrt[3]{a^3+26}\overset{Lema}{\geq}\sum\frac{a+26}{9}=\frac{\sum a+78}{9}=\frac{1}{9}\sum a+\frac{78}{9}\overset{CS}{\geq}\frac{1}{9}\cdot\frac{9}{\sum\frac{1}{a}}+\frac{78}{9}\Rightarrow$$

$$\Rightarrow 9\geq\frac{1}{9}\cdot\frac{9}{\sum\frac{1}{a}}+\frac{78}{9}\Rightarrow\sum\frac{1}{a}\geq 3.$$

Egalitatea are loc daca si numai daca $a=b=c=1$.

Problema121.

In ΔABC

$$\sum r_a\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\leq\frac{9R}{2}.$$

RMM5/2024, Nguyen Hung Cuong, Vietnam

Soluție.

Tripletele (r_a,r_b,r_c) și $\left(\sin\frac{B}{2}+\sin\frac{C}{2},\sin\frac{C}{2}+\sin\frac{A}{2},\sin\frac{A}{2}+\sin\frac{B}{2}\right)$ sunt invers ordonate.

$$\sum r_a\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\overset{Chebyshev}{\leq}\frac{1}{3}\sum r_a\sum\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\leq\frac{1}{3}\cdot\frac{9R}{2}\cdot 3=\frac{9R}{2}.$$

Am folosit mai sus $\sum r_a\leq\frac{9R}{2}$ și $\sum\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)=2\sum\sin\frac{A}{2}\overset{Jensen}{\leq}2\cdot\frac{3}{2}=3$.

Egalitatea are loc daca si numai daca triunghiul este echilateral..

Remarca.

In ΔABC

$$\sum r_a\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\geq 9r\left(\frac{2r}{R}\right)^{\frac{1}{3}}.$$

Marin Chirciu

Soluție.

$$\sum r_a\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\overset{AM-GM}{\geq}3\sqrt[3]{\prod r_a\prod\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}\geq 3\sqrt[3]{27r^3\cdot\frac{2r}{R}}=9r\left(\frac{2r}{R}\right)^{\frac{1}{3}}.$$

Am folosit mai sus $\prod r_a\geq 27r^3$ și $\prod\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\geq\frac{2r}{R}$, vezi

$$\prod\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\geq\prod 2\sqrt{\sin\frac{B}{2}\cdot\sin\frac{C}{2}}=8\sum\sin\frac{A}{2}=8\cdot\frac{r}{4R}=\frac{2r}{R}.$$

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Remarca.

In ΔABC

$$1).\ 9r\left(\frac{2r}{R}\right)^{\frac{1}{3}}\leq\sum r_a\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\leq\frac{9R}{2}.$$

$$2).\ \sum h_a\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\geq 9r\left(\frac{2r}{R}\right)^{\frac{1}{3}}.$$

$$3).\ \sum m_a\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\geq 9r\left(\frac{2r}{R}\right)^{\frac{1}{3}}.$$

$$4).\ \sum w_a\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\geq 9r\left(\frac{2r}{R}\right)^{\frac{1}{3}}.$$

$$5).\ \sum s_a\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\geq 9r\left(\frac{2r}{R}\right)^{\frac{1}{3}}.$$

Dezvoltări, Marin Chirciu

Remarca.

In ΔABC

$$\frac{2}{R}\left(\frac{2r}{R}\right)^{\frac{1}{3}}\leq\sum\frac{1}{h_a}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\leq\frac{1}{r}.$$

Marin Chirciu

Soluție.

Inegalitatea din dreapta.

Tripletele $\left(\frac{1}{h_a},\frac{1}{h_b},\frac{1}{h_c}\right)$ și $\left(\sin\frac{B}{2}+\sin\frac{C}{2},\sin\frac{C}{2}+\sin\frac{A}{2},\sin\frac{A}{2}+\sin\frac{B}{2}\right)$ sunt invers ordonate.

Folosind inegalitatea lui Chebyshev obținem:

$$\sum\frac{1}{h_a}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\overset{Chebyshev}{\leq}\frac{1}{3}\sum\frac{1}{h_a}\sum\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\leq\frac{1}{3}\cdot\frac{1}{r}\cdot 3=\frac{1}{r}.$$

Am folosit mai sus $\sum\frac{1}{h_a}=\frac{1}{r}$ și $\sum\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)=2\sum\sin\frac{A}{2}\overset{Jensen}{\leq}2\cdot\frac{3}{2}=3$.

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Inegalitatea din stânga.

$$\sum\frac{1}{h_a}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{1}{h_a}\prod\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}\geq 3\sqrt[3]{\left(\frac{2}{3R}\right)^3\cdot\frac{2r}{R}}=3\cdot\frac{2}{3R}\left(\frac{2r}{R}\right)^{\frac{1}{3}}=$$

$=\frac{2}{R}\left(\frac{2r}{R}\right)^{\frac{1}{3}}.$

Am folosit mai sus $\prod h_a \le \left(\frac{3R}{2}\right)^3$ și $\prod\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\ge\frac{2r}{R}$,vezi

$\prod\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\ge\prod 2\sqrt{\sin\frac{B}{2}\cdot\sin\frac{C}{2}}=8\sum\sin\frac{A}{2}=8\cdot\frac{r}{4R}=\frac{2r}{R}.$

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Remarca.

In ΔABC

$$1).\ \frac{2}{R}\left(\frac{2r}{R}\right)^{\frac{1}{3}}\le\sum\frac{1}{w_a}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\le\frac{1}{r}.$$

$$2).\ \frac{2}{R}\left(\frac{2r}{R}\right)^{\frac{1}{3}}\le\sum\frac{1}{m_a}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\le\frac{1}{r}.$$

$$3).\ \frac{2}{R}\left(\frac{2r}{R}\right)^{\frac{1}{3}}\le\sum\frac{1}{s_a}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\le\frac{1}{r}.$$

Dezvoltări, Marin Chirciu

Problema122.

If $x,y,z>0$, $x+y+z=3$ then

$$\frac{1}{x^2+y^2+z^2}+\frac{2009}{xy+yz+zx}\ge 670.$$

THCS 5/2024, Pham Van Tuyen, Vietnam

Soluție.

1). $xy+yz+zx\le 3$, vezi: $xy+yz+zx\le\frac{(x+y+z)^2}{3}=\frac{3^2}{3}=3.$

2). $\frac{1}{x^2+y^2+z^2}+\frac{2}{xy+yz+zx}\ge 1,$

vezi $\frac{1}{\sum x^2}+\frac{2}{\sum xy}=\frac{1}{\sum x^2}+\frac{1}{\sum xy}+\frac{1}{\sum xy}\overset{CS}{\ge}\frac{(1+1+1)^2}{\sum x^2+2\sum xy}=\frac{9}{\left(\sum x\right)^2}=\frac{9}{3^2}=1.$

Folosind 1) și 2) obținem:

$LHS=\frac{1}{\sum x^2}+\frac{2009}{\sum xy}\overset{2)}{\ge}\left(1-\frac{2}{\sum xy}\right)+\frac{2009}{\sum xy}=1+\frac{2007}{\sum xy}\overset{1)}{\ge}1+\frac{2007}{3}=1+669=670=RHS.$

Egalitatea are loc daca si numai daca $x=y=z=1$.

Remarca.

If $x,y,z>0$, $x+y+z=3$ and $\lambda\ge 2$ then

$$\frac{1}{x^2+y^2+z^2}+\frac{\lambda}{xy+yz+zx}\ge\frac{\lambda+1}{3}.$$

Marin Chirciu

Problema123.

If $a,b\ge 0$, $a+b=1$ then

$$\frac{a+1}{a^2+b}+\frac{b+1}{b^2+a}\ge 3.$$

RMM5/2024, Nguyen Hung Cuong, Vietnam

Soluție.

Lema.

If $a,b\ge 0$, $a+b=1$ then

$$\frac{a+1}{a^2+b}\ge a+1.$$

$a+b=1\Leftrightarrow b=1-a\ge 0\Rightarrow a(1-a)\ge 0\Rightarrow a^2\le a\Rightarrow a^2+b\le a+b=1\Rightarrow a^2+b\le 1\Rightarrow$

$\Rightarrow \frac{a+1}{a^2+b}\ge a+1$, cu egalitate pentru $(a,b)=(1,0)$ sau $(a,b)=(0,1)$.

$2HS=\frac{a+1}{a^2+b}+\frac{b+1}{b^2+a}\overset{Lema}{\ge}(a+1)+(b+1)=3=RHS$

Egalitatea are loc daca si numai daca $(a,b)=(1,0)$ sau $(a,b)=(0,1)$.

Remarca.

If $a,b\ge 0$, $a+b=\lambda$, $\lambda>0$ then

$$\frac{a+1}{a^2+\lambda b}+\frac{b+1}{b^2+\lambda a}\ge\frac{\lambda+2}{\lambda^2}.$$

Marin Chirciu

Soluție.

Lema.

If $a,b\ge 0$, $a+b=\lambda$, $\lambda>0$ then

$$\frac{a+1}{a^2+\lambda b}\ge\frac{a+1}{\lambda^2}.$$

$a+b=\lambda\Leftrightarrow b=\lambda-a\ge 0\Rightarrow$

$a(\lambda-a)\ge 0\Rightarrow a^2\le\lambda a\Rightarrow a^2+\lambda b\le\lambda a+\lambda b=\lambda^2\Rightarrow a^2+\lambda b\le\lambda^2\Rightarrow$

$\Rightarrow\frac{a+1}{a^2+\lambda b}\ge\frac{a+1}{\lambda^2}$, cu egalitate pentru $(a,b)=(\lambda,0)$ sau $(a,b)=(0,\lambda)$.

$2HS=\frac{a+1}{a^2+\lambda b}+\frac{b+1}{b^2+\lambda a}\overset{Lema}{\ge}\frac{a+1}{\lambda^2}+\frac{a+1}{\lambda^2}=\frac{\lambda+2}{\lambda^2}=RHS$

Egalitatea are loc daca si numai daca $(a,b)=(\lambda,0)$ sau $(a,b)=(0,\lambda)$.

Problema124.

If $a,b,c>0$ then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{6}{a+b+c}\ge\frac{15}{abc+2}.$$

Mathematical Inequalites 5/2024, Nguyen Viet Hung, Vietnam

Soluție.

Punând $abc=t^3$ și $\left(\frac{a}{t},\frac{b}{t},\frac{c}{t}\right)=(x,y,z)\Rightarrow xyz=1$ și inegalitatea se scrie:

$\frac{1}{tx}+\frac{1}{ty}+\frac{1}{tz}+\frac{6}{tx+ty+tz}\ge\frac{15}{t^3+2}\Leftrightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{6}{x+y+z}\ge\frac{15t}{t^3+2}$, care rezultă din:

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{6}{x+y+z}\ge 5\ge\frac{15t}{t^3+2}$.

Egalitatea are loc daca si numai daca $a=b=c=1$.

Remarca.

If $a,b,c>0$ and $\lambda\ge\frac{9}{2}$ then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{\lambda}{a+b+c}\ge\frac{\lambda+9}{abc+2}.$$

Marin Chirciu

Soluție.

Punând $abc=t^3$ și $\left(\frac{a}{t},\frac{b}{t},\frac{c}{t}\right)=(x,y,z)\Rightarrow xyz=1$ și inegalitatea se scrie:

$\frac{1}{tx}+\frac{1}{ty}+\frac{1}{tz}+\frac{\lambda}{tx+ty+tz}\ge\frac{\lambda+9}{t^3+2}\Leftrightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{\lambda}{x+y+z}\ge\frac{(\lambda+9)t}{t^3+2}$, care rezultă din:

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{\lambda}{x+y+z}\overset{(1)}{\ge}3+\frac{\lambda}{3}\overset{(2)}{\ge}\frac{(\lambda+9)t}{t^3+2}$.

(1) $\Leftrightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{\lambda}{x+y+z}\ge3+\frac{\lambda}{3}\Leftrightarrow\frac{xy+yz+zx}{xyz}+\frac{\lambda}{x+y+z}\ge\frac{\lambda+9}{3}\Leftrightarrow$

$\overset{xyz=1}{\Leftrightarrow}\frac{xy+yz+zx}{1}+\frac{\lambda}{x+y+z}\ge\frac{\lambda+9}{3}\overset{pqr-Method}{\Leftrightarrow}q+\frac{\lambda}{p}\ge\frac{\lambda+9}{3}$.

Avem $q^2=(xy+yz+zx)^2\ge3xyz(x+y+z)=3rp\overset{r=1}{\Rightarrow}q^2\ge3p\Rightarrow q\ge\sqrt{3p}=t$.

Este suficient să arătăm că:

$t+\frac{\lambda}{\frac{t^2}{3}}\ge\frac{\lambda+9}{3}\Leftrightarrow t+\frac{3\lambda}{t^2}\ge\frac{\lambda+9}{3}\Leftrightarrow3t^3-(\lambda+9)t^2+9\lambda\ge0\Leftrightarrow(t-3)(3t^2-\lambda t-3\lambda)\ge0$,

vezi $t\ge3$, adevărată din $t=\sqrt{3p}$, $p=x+y+z\ge3\sqrt[3]{xyz}=3$.

$(3t^2-\lambda t-3\lambda)>0$, pentru $\lambda\ge\frac{9}{2}$.

(2) $\Leftrightarrow3+\frac{\lambda}{3}\ge\frac{(\lambda+9)t}{t^3+2}\Leftrightarrow t^3+2\ge3t\Leftrightarrow t^3-3t+2\ge0\Leftrightarrow(t-1)^2(t+2)\ge0$.

Egalitatea are loc daca si numai daca $a=b=c=1$.

Problema125.

In acute ΔABC

$$\sum\frac{\sec B+\sec C}{\sec^2 A}\ge3.$$

Mathematical Inequalities5/2024, George Apostolopoulos, Greece

Soluție.

Lema.

In acute ΔABC

$$\sum\frac{\sec B+\sec C}{\sec^2 A}=\frac{p^2(8R^2+2Rr-p^2)-16R^4+12R^2r^2+6Rr^3+r^4}{2R^2\left[p^2-(2R+r)^2\right]}.$$

Folosind **Lema** inegalitatea se scrie:

$$\frac{p^2(8R^2+2Rr-p^2)-16R^4+12R^2r^2+6Rr^3+r^4}{2R^2\left[p^2-(2R+r)^2\right]}\ge3\Leftrightarrow$$

$$\Leftrightarrow p^2(2R^2+2Rr-p^2)+8R^4+24R^3r+18R^2r^2+6Rr^3+r^4\ge0.$$

Distingem cazurile:

Cazul1). Dacă $(2R^2+2Rr-p^2)\ge0$, inegalitatea este evidentă.

Cazul2). Dacă $(2R^2+2Rr-p^2)<0$, inegalitatea se rescrie:

$8R^4+24R^3r+18R^2r^2+6Rr^3+r^4\ge p^2\left(p^2-2R^2-2Rr\right)$, care rezultă din inegalitatea lui Gerretsen: $p^2\le 4R^2+4Rr+3r^2$.

Rămâne să arătăm că:

$8R^4+24R^3r+18R^2r^2+6Rr^3+r^4\ge\left(4R^2+4Rr+3r^2\right)\left(4R^2+4Rr+3r^2-2R^2-2Rr\right)\Leftrightarrow$

$\Leftrightarrow 8R^4+24R^3r+18R^2r^2+6Rr^3+r^4\ge\left(4R^2+4Rr+3r^2\right)\left(2R^2+2Rr+3r^2\right)\Leftrightarrow$

$\Leftrightarrow 8R^4+24R^3r+18R^2r^2+6Rr^3+r^4\ge\left(4R^2+4Rr+3r^2\right)\left(2R^2+2Rr+3r^2\right)\Leftrightarrow$

$\Leftrightarrow 8R^4+24R^3r+18R^2r^2+6Rr^3+r^4\ge\left(4R^2+4Rr+3r^2\right)\left(2R^2+2Rr+3r^2\right)\Leftrightarrow$

$\Leftrightarrow 2R^3-2R^2r-3Rr^2-2r^3\ge 0\Leftrightarrow (R-2r)\left(2R^2+2Rr+r^2\right)\ge 0$, vezi $R\ge 2r$,(Euler).

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Remarca.

In acute ΔABC

$$\sum\frac{\sec B+\sec C}{\sec^2 A}\le\frac{\left(\frac{R}{r}\right)^4-10}{10-2\left(\frac{R}{r}\right)^2}.$$

Marin Chirciu

Soluție.

Lema.

In acute ΔABC

$$\sum\frac{\sec B+\sec C}{\sec^2 A}=\frac{p^2\left(8R^2+2Rr-p^2\right)-16R^4+12R^2r^2+6Rr^3+r^4}{2R^2\left[p^2-(2R+r)^2\right]}.$$

Folosind **Lema** inegalitatea se scrie:

$$\frac{p^2\left(8R^2+2Rr-p^2\right)-16R^4+12R^2r^2+6Rr^3+r^4}{2R^2\left[p^2-(2R+r)^2\right]}\overset{Walker}{\le}$$

$$\overset{Walker}{\le}\frac{p^2\left(8R^2+2Rr-2R^2-8Rr-3r^2\right)-16R^4+12R^2r^2+6Rr^3+r^4}{2R^2\left[2R^2+8Rr+3r^2-(2R+r)^2\right]}=$$

$$=\frac{p^2\left(6R^2-6Rr-3r^2\right)-16R^4+12R^2r^2+6Rr^3+r^4}{2R^2\left(-2R^2+4Rr+2r^2\right)}\overset{Walker}{\le}$$

$$\overset{Walker}{\le}\frac{\left(2R^2+8Rr+3r^2\right)\left(6R^2-6Rr-3r^2\right)-16R^4+12R^2r^2+6Rr^3+r^4}{2R^2\left(-2R^2+4Rr+2r^2\right)}=$$

$$=\frac{8R^4-6R^2r^2-24Rr^3-8r^4}{4R^2\left(-R^2+2Rr+r^2\right)}=\frac{4R^4-3R^2r^2-12Rr^3-4r^4}{2R^2\left(-R^2+2Rr+r^2\right)}\overset{Euler}{\le}\frac{4R^4-40r^4}{8r^2\left(-R^2+5r^2\right)}=$$

$$=\frac{R^4-10r^4}{2r^2\left(5r^2-R^2\right)}=\frac{\left(\frac{R}{r}\right)^4-10}{10-2\left(\frac{R}{r}\right)^2}.$$

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Remarca.

In acute ΔABC

$$3\le\sum\frac{\sec B+\sec C}{\sec^2 A}\le\frac{\left(\frac{R}{r}\right)^4-10}{10-2\left(\frac{R}{r}\right)^2}.$$

Remarca.
In ΔABC

$$\frac{27r^2}{F}\le\sum\frac{\csc B+\csc C}{\csc^2 A}\le\frac{27R^2}{4F}.$$

Marin Chirciu

Soluție.
Lema.
In ΔABC

$$\sum\frac{\csc B+\csc C}{\csc^2 A}=\frac{p^2\left(p^2-4Rr\right)-r^2\left(4R+r\right)^2}{4R^2rp}.$$

Inegalitatea din dreapta.

$$\sum\frac{\csc B+\csc C}{\csc^2 A}=\frac{p^2\left(p^2-4Rr\right)-r^2\left(4R+r\right)^2}{4R^2rp}\overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{\left(4R^2+4Rr+3r^2\right)\left(4R^2+4Rr+3r^2-4Rr\right)-r^2\left(4R+r\right)^2}{4R^2rp}=$$

$$=\frac{\left(4R^2+4Rr+3r^2\right)\left(4R^2+3r^2\right)-r^2\left(4R+r\right)^2}{4R^2rp}=\frac{4R^4+4R^3r+2R^2r^2+Rr^3+2r^4}{R^2rp}\overset{Euler}{\le}$$

$$\overset{Euler}{\le}\frac{\frac{27R^4}{4}}{R^2rp}=\frac{27R^2}{4rp}.$$

Egalitatea are loc daca si numai daca triunghiul este echilateral.
Inegalitatea din stânga.

$$\sum\frac{\csc B+\csc C}{\csc^2 A}=\frac{p^2\left(p^2-4Rr\right)-r^2\left(4R+r\right)^2}{4R^2rp}\overset{Gerretsen}{\ge}$$

$$\overset{Gerretsen}{\ge}\frac{\left(16Rr-5r^2\right)\left(16Rr-5r^2-4Rr\right)-r^2\left(4R+r\right)^2}{4R^2rp}=\frac{r^2\left(44R^2-37Rr+6r^2\right)}{R^2rp}\overset{Euler}{\ge}$$

$$\overset{Euler}{\ge}\frac{r^2\cdot 27R^2}{R^2rp}=\frac{27r^2}{rp}.$$

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Remarca.
In acute ΔABC

$$\sqrt{3}\cdot\sum\frac{\sec B+\sec C}{\sec^2 A}-\sum\frac{\csc B+\csc C}{\csc^2 A}\ge\frac{20}{F}\left(\frac{16r^4}{R^2}-R^2\right).$$

Marin Chirciu

Soluție.
Folosind **Lemele** de mai sus, obținem:

$$\sqrt{3}\cdot\frac{p^2\left(8R^2+2Rr-p^2\right)-16R^4+12R^2r^2+6Rr^3+r^4}{2R^2\left[p^2-\left(2R+r\right)^2\right]}-\frac{p^2\left(p^2-4Rr\right)-r^2\left(4R+r\right)^2}{4R^2rp}\overset{Gerretsen}{\ge}$$

$$\overset{Gerretsen}{\geq}\sqrt{3}\cdot\frac{p^2\left(8R^2+2Rr-p^2\right)-16R^4+12R^2r^2+6Rr^3+r^4}{2R^2\left[4R^2+4Rr+3r^2-(2R+r)^2\right]}-\frac{p^2\left(p^2-4Rr\right)-r^2(4R+r)^2}{4R^2rp}=$$

$$\overset{Gerretsen}{\geq}\sqrt{3}\cdot\frac{p^2\left(8R^2+2Rr-p^2\right)-16R^4+12R^2r^2+6Rr^3+r^4}{2R^2\left[4R^2+4Rr+3r^2-(2R+r)^2\right]}-\frac{p^2\left(p^2-4Rr\right)-r^2(4R+r)^2}{4R^2rp}=$$

$$=\sqrt{3}\cdot\frac{p^2\left(8R^2+2Rr-p^2\right)-16R^4+12R^2r^2+6Rr^3+r^4}{2R^2\cdot 2r^2}-\frac{p^2\left(p^2-4Rr\right)-r^2(4R+r)^2}{4R^2rp}=$$

$$=p\sqrt{3}\cdot\frac{p^2\left(8R^2+2Rr-p^2\right)-16R^4+12R^2r^2+6Rr^3+r^4}{4R^2r^2p}-\frac{p^2\left(p^2-4Rr\right)-r^2(4R+r)^2}{4R^2rp}\overset{p\geq 3r\sqrt{3}}{\geq}$$

$$\overset{p\geq 3r\sqrt{3}}{\geq}3r\sqrt{3}\cdot\sqrt{3}\cdot\frac{p^2\left(8R^2+2Rr-p^2\right)-16R^4+12R^2r^2+6Rr^3+r^4}{4R^2r^2p}-\frac{p^2\left(p^2-4Rr\right)-r^2(4R+r)^2}{4R^2rp}=$$

$$=9\cdot\frac{p^2\left(8R^2+2Rr-p^2\right)-16R^4+12R^2r^2+6Rr^3+r^4}{4R^2rp}-\frac{p^2\left(p^2-4Rr\right)-r^2(4R+r)^2}{4R^2rp}=$$

$$=\frac{9\left[p^2\left(8R^2+2Rr-p^2\right)-16R^4+12R^2r^2+6Rr^3+r^4\right]-p^2\left(p^2-4Rr\right)+r^2(4R+r)^2}{4R^2rp}=$$

$$=\frac{p^2\left(72R^2+22Rr-10p^2\right)-144R^4+124R^2r^2+62Rr^3+10r^4}{4R^2F}\overset{Walker}{\geq}$$

$$\overset{Walker}{\geq}\frac{\left(2R^2+8Rr+3r^2\right)\left(36R^2+11Rr-5\left(4R^2+4Rr+3r^2\right)\right)-72R^4+62R^2r^2+31Rr^3+5r^4}{2R^2F}=$$

$$=\frac{-40R^4+110R^3r+8R^2r^2-116Rr^3-40r^4}{2R^2F}=\frac{-20R^4+55R^3r+4R^2r^2-58Rr^3-20r^4}{R^2F}\overset{Euler}{\geq}$$

$$\overset{Euler}{\geq}\frac{-20R^4+320r^4}{R^2F}=\frac{20}{F}\left(\frac{16r^4}{R^2}-R^2\right).$$

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Remarca.

In acute ΔABC

$$\sqrt{3}\cdot\sum\frac{\sec B+\sec C}{\sec^2 A}\geq\sum\frac{\csc B+\csc C}{\csc^2 A}+\frac{20}{F}\left(\frac{16r^4}{R^2}-R^2\right).$$

Marin Chirciu

Problema126.

If $a,b,c>0$ then

$$\sqrt[3]{abc}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 2\sqrt{3}.$$

Math Olymp5/2024, Elton Papanikolla

Soluție.

Folosind inegalitatea mediilor obținem: $\frac{\sqrt[3]{abc}}{3}+\frac{1}{a}\geq\frac{2}{\sqrt{3}}\sqrt[6]{\frac{bc}{a^2}}$, cu egalitate pentru $\frac{\sqrt[3]{abc}}{3}=\frac{1}{a}$.

$$LHS=\sum\left(\frac{\sqrt[3]{abc}}{3}+\frac{1}{a}\right)\geq\sum\frac{2}{\sqrt{3}}\sqrt[6]{\frac{bc}{a^2}}\overset{AM-GM}{\geq}\frac{2}{\sqrt{3}}\cdot 3\sqrt[3]{\prod\sqrt[6]{\frac{bc}{a^2}}}=\frac{2}{\sqrt{3}}\cdot 3\cdot 1=2\sqrt{3}=RHS.$$

Egalitatea are loc daca si numai daca $\frac{1}{a}=\frac{1}{b}=\frac{1}{c}=\frac{\sqrt[3]{abc}}{3} \Leftrightarrow a=b=c=\sqrt{3}$.

Remarca.

If $a,b,c,d>0$ then

$$\sqrt[4]{abcd}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\geq 4.$$

Marin Chirciu

Soluție.

Folosind inegalitatea mediilor obținem: $\frac{\sqrt[4]{abcd}}{4}+\frac{1}{a}\geq\sqrt[8]{\frac{bcd}{a^3}}$, cu egalitate pentru $\frac{\sqrt[4]{abcd}}{4}=\frac{1}{a}$.

$$LHS=\sum\left(\frac{\sqrt[4]{abc}}{4}+\frac{1}{a}\right)\geq\sum\sqrt[8]{\frac{bcd}{a^3}}\overset{AM-GM}{\geq}4\sqrt[4]{\prod\sqrt[8]{\frac{bcd}{a^3}}}=4\cdot 1=4=RHS.$$

Egalitatea are loc daca si numai daca $\frac{1}{a}=\frac{1}{b}=\frac{1}{c}=\frac{1}{d}=\frac{\sqrt[4]{abcd}}{4} \Leftrightarrow a=b=c=d=2$.

Problema127.

If $a,b>0$ then

$$\sqrt{3a^2+1}+\sqrt{3b^2+1}+6\left(\frac{1}{a+1}+\frac{1}{b+1}\right)\geq 10.$$

RMM5/2024, Nguyen Hung Cuong, Vietnam

Soluție.

Lema.

If $a,b>0$ then

$$\sqrt{3a^2+1}+\frac{6}{a+1}\geq 5.$$

Demonstrație.

$$\sqrt{3a^2+1}+\frac{6}{a+1}\geq 5 \Leftrightarrow \sqrt{3a^2+1}+\frac{6}{a+1}\geq 5 \Leftrightarrow \sqrt{3a^2+1}\geq\frac{5a-1}{a+1} \Leftrightarrow$$

$$\Leftrightarrow (a+1)^2\left(3a^2+1\right)\geq(5a-1)^2 \Leftrightarrow 3a^4+6a^3-21a^2+12a\geq 0 \Leftrightarrow 3a(a-1)^2(a+4)\geq 0.$$

$$LHS=\sqrt{3a^2+1}+\sqrt{3b^2+1}+6\left(\frac{1}{a+1}+\frac{1}{b+1}\right)\overset{Lema}{\geq}5+5=10=RHS.$$

Egalitatea are loc daca si numai daca $a=b=1$.

Remarca.

If $a,b,c>0$ then

$$\sqrt{3a^2+1}+\sqrt{3b^2+1}+\sqrt{3c^2+1}+6\left(\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\right)\geq 15.$$

Marin Chirciu

Soluție.

Lema.

If $a>0$ then

$$\sqrt{3a^2+1}+\frac{6}{a+1}\geq 5.$$

$$LHS=\sum\left(\sqrt{3a^2+1}+\frac{6}{a+1}\right)\overset{Lema}{\geq}\sum 5=15=RHS.$$

Egalitatea are loc daca si numai daca $a=b=c=1$.

Remarca.

If $a,b\ge\frac{1}{6}$ then

$$\sqrt{8a^2+1}+\sqrt{8b^2+1}+\frac{32}{3}\left(\frac{1}{a+1}+\frac{1}{b+1}\right)\ge\frac{50}{3}.$$

Marin Chirciu

Soluție.

Lema.

If $a\ge\frac{1}{6}$ then

$$\sqrt{8a^2+1}+\frac{32}{3(a+1)}\ge\frac{25}{3}.$$

Demonstrație.

$\sqrt{8a^2+1}+\frac{32}{3(a+1)}\ge\frac{25}{3}\Leftrightarrow\Leftrightarrow\sqrt{8a^2+1}\ge\frac{25a-7}{3(a+1)}\Leftrightarrow$

$\Leftrightarrow 9(a+1)^2(8a^2+1)\ge(25a-7)^2\Leftrightarrow 9a^4+18a^3-68a^2+46a-5\ge0\Leftrightarrow$

$\Leftrightarrow (a-1)^2(9a^2+36a-5)\ge0$, vezi $a\ge\frac{1}{6}$, care asigură $(9a^2+36a-5)>0$.

$LHS=\sum\left(\sqrt{8a^2+1}+\frac{32}{3}\frac{1}{a+1}\right)\overset{Lema}{\ge}\sum\frac{25}{3}=\frac{50}{3}=RHS$.

Egalitatea are loc daca si numai daca $a=b=1$.

Remarca.

If $a,b,c\ge\frac{1}{6}$ then

$$\sqrt{8a^2+1}+\sqrt{8b^2+1}+\sqrt{8c^2+1}+\frac{32}{3}\left(\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\right)\ge25.$$

Marin Chirciu

Problema128.

If $x\ge1$ then

$$x^5-\frac{1}{x^4}\ge9(x-1).$$

Mathematics(College and High Scholl)5/2024, Amir Sofi, Kosovo

Soluție.

$x^5-\frac{1}{x^4}\ge9(x-1)\Leftrightarrow x^9-1\ge9x^4(x-1)$,

vezi $x\ge1$ și $x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1\ge9x^4$, care rezultă din inegalitatea mediilor:

$x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1\ge9\sqrt[9]{x^8\cdot x^7\cdot x^6\cdot x^5\cdot x^4\cdot x^3\cdot x^2\cdot x\cdot1}=9\sqrt[9]{x^{36}}=9x^4$.

Egalitatea are loc daca si numai daca $x=1$.

Remarca.

If $x\ge1$ and $n\in\mathbf{N}$ then

$$x^{n+1}-\frac{1}{x^n}\ge(2n+1)(x-1).$$

Marin Chirciu

Soluție.

Pentru $n=0$ se obține egalitatea $x-1=x-1$.

În continuare fie $n\in\mathbf{N}^*$.

$x^{n+1}-\frac{1}{x^n}\geq(2n+1)(x-1)\Leftrightarrow x^{2n+1}-1\geq(2n+1)x^n(x-1)$,

vezi $x\geq 1$ și $x^{2n+1}+x^{2n}+x^{2n-1}+x^{2n-2}+...+x^2+x+1\geq(2n+1)x^n$, care rezultă din AM-G:

$x^{2n+1}+x^{2n}+x^{2n-1}+x^{2n-2}+...+x^2+x+1\geq(2n+1)\sqrt[2n+1]{x^{2n+1}\cdot x^{2n}\cdot x^{2n-1}\cdot x^{2n-2}\cdot...\cdot x^2\cdot x\cdot 1}=$

$=(2n+1)\sqrt[2n+1]{x^{n(2n+1)}}=(2n+1)x^n$.

Egalitatea are loc daca si numai daca $x=1$.

Problema129.

If $a,b>0$ then

$$\frac{a^3}{b}+\frac{b^3}{a}+2\left(\frac{a}{b^2}+\frac{b}{a^2}\right)\geq 6.$$

RMM 5/2024, Nguyen Hung Cuong, Vietnam

Soluție.

$LHS=\frac{a^3}{b}+\frac{b^3}{a}+2\left(\frac{a}{b^2}+\frac{b}{a^2}\right)\overset{AM-GM}{\geq}2\sqrt{\frac{a^3}{b}\cdot\frac{b^3}{a}}+2\left(\frac{a}{b^2}+\frac{b}{a^2}\right)=2ab+2\left(\frac{a}{b^2}+\frac{b}{a^2}\right)=$

$=2\left(ab+\frac{a}{b^2}+\frac{b}{a^2}\right)\overset{AM-GM}{\geq}2\cdot 3\sqrt[3]{ab\cdot\frac{a}{b^2}\cdot\frac{b}{a^2}}=6=RHS$.

Egalitatea are loc daca si numai daca $a=b=1$.

Remarca.

If $a,b>0$ then

$$\frac{a^5}{b}+\frac{b^5}{a}+2\left(\frac{a}{b^3}+\frac{b}{a^3}\right)\geq 6.$$

Remarca.

If $a,b>0$ and $n\in\mathbf{N}$ then

$$\frac{a^{2n+3}}{b}+\frac{b^{2n+3}}{a}+2\left(\frac{a}{b^{2n+1}}+\frac{b}{a^{2n+1}}\right)\geq 6.$$

Marin Chirciu

Soluție.

$LHS=\frac{a^{2n+3}}{b}+\frac{b^{2n+3}}{a}+2\left(\frac{a}{b^{2n+1}}+\frac{b}{a^{2n+1}}\right)\overset{AM-GM}{\geq}2\sqrt{\frac{a^{2n+3}}{b}\cdot\frac{b^{2n+3}}{a}}+2\left(\frac{a}{b^{2n+1}}+\frac{b}{a^{2n+1}}\right)=$

$=2a^{n+1}b^{n+1}+2\left(\frac{a}{b^{2n+1}}+\frac{b}{a^{2n+1}}\right)=2\left(a^{n+1}b^{n+1}+\frac{a}{b^{2n+1}}+\frac{b}{a^{2n+1}}\right)\overset{AM-GM}{\geq}2\cdot 3\sqrt[3]{a^{n+1}b^{n+1}\cdot\frac{a}{b^{2n+1}}\cdot\frac{b}{a^{2n+1}}}=$

$=6=RHS$.

Egalitatea are loc daca si numai daca $a=b=1$.

Remarca.

If $a,b,c>0$ and $n\in\mathbf{N}$ then

$$\frac{a^{2n+3}}{b}+\frac{b^{2n+3}}{c}+\frac{c^{2n+3}}{a}+3\left(\frac{a}{b^{n+2}}+\frac{b}{c^{n+2}}+\frac{c}{a^{n+2}}\right)\geq 12.$$

Marin Chirciu

Problema130.

If $x,y,z>0$, $x+y+z=3$ then

$$\sum\frac{(x+1)^2}{(x+y)(x+z)}\geq 6.$$

RMM 5/2024, Bui Duc Cuong, Vietnam

Soluție.
Lema.
If $x, y, z > 0$, $x+y+z=3$ then

$$\frac{(x+1)^2}{(x+y)(x+z)} \geq x-y-z+3.$$

Demonstrație.

$$\frac{(x+1)^3}{(x+y)(x+z)}+(x+y)+(x+z) \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{(x+1)^3}{(x+y)(x+z)}\cdot(x+y)\cdot(x+z)}=3(x+1),$$

cu egalitate pentru $\frac{(x+1)^3}{(x+y)(x+z)}=(x+y)=(x+z) \Leftrightarrow y=z=1$.

Rezultă $\frac{(x+1)^3}{(x+y)(x+z)}+(x+y)+(x+z)\geq 3(x+1) \Leftrightarrow \frac{(x+1)^2}{(x+y)(x+z)} \geq x-y-z+3$.

$$LHS=\sum\frac{(x+1)^2}{(x+y)(x+z)} \overset{Lema}{\geq} \sum(x-y-z+3)=9-\sum x=9-3=6=RHS$$

$$LHS=\frac{a^3}{b}+\frac{b^3}{a}+2\left(\frac{a}{b^2}+\frac{b}{a^2}\right) \overset{AM-GM}{\geq} 2\sqrt{\frac{a^3}{b}\cdot\frac{b^3}{a}}+2\left(\frac{a}{b^2}+\frac{b}{a^2}\right)=2ab+2\left(\frac{a}{b^2}+\frac{b}{a^2}\right)=$$

$$=2\left(ab+\frac{a}{b^2}+\frac{b}{a^2}\right) \overset{AM-GM}{\geq} 2\cdot 3\sqrt[3]{ab\cdot\frac{a}{b^2}\cdot\frac{b}{a^2}}=6=RHS.$$

Egalitatea are loc daca si numai daca $x=y=z=1$.

Remarca.
In ΔABC

$$\sum\frac{r_b r_c(3r+r_a)^3}{r_a(r_a+r_b)(r_a+r_c)} \geq 54r^2.$$

Soluție.
Lema.
If $x, y, z > 0$, $x+y+z=3$ then

$$\sum\frac{(x+1)^3}{(x+y)(x+z)} \geq 6.$$

Se cunoaste identitatea in triunghi $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r} \Leftrightarrow \frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$.

Folosind **Lema** pentru $(x,y,z)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ obtinem:

$$\sum\frac{\left(\frac{3r}{r_a}+1\right)^3}{\left(\frac{3r}{r_a}+\frac{3r}{r_b}\right)\left(\frac{3r}{r_a}+\frac{3r}{r_c}\right)} \geq 6 \Leftrightarrow \sum\frac{r_b r_c(3r+r_a)^3}{r_a(r_a+r_b)(r_a+r_c)} \geq 54r^2.$$

Egalitatea are loc daca si numai daca triunghiul este echilateral.

Remarca.
In ΔABC

$$\sum \frac{h_b h_c (3r+h_a)^3}{h_a(h_a+h_b)(h_a+h_c)} \geq 54r^2.$$

Marin Chirciu

Problema131.

1) If $a,b,c>0$ then

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}.$$

USAMO-1997

Soluție.

Folosind inegalitatea $x^3+y^3 \geq xy(x+y) \Leftrightarrow (x+y)(x-y)^2 \geq 0$, $x,y>0$ obținem:

$$Ms = \sum \frac{1}{b^3+c^3+abc} \leq \sum \frac{1}{bc(b+c)+abc} = \sum \frac{1}{bc(a+b+c)} = \frac{1}{abc} = Md.$$

Egalitatea are loc dacă și numai dacă $a=b=c$.

Remarca.

2) If $a,b,c>0$ and $\lambda \geq 1$ then

$$\frac{1}{a^3+b^3+\lambda abc}+\frac{1}{b^3+c^3+\lambda abc}+\frac{1}{c^3+a^3+\lambda abc} \leq \frac{3}{(\lambda+2)abc}.$$

Marin Chirciu

Soluție.

Folosind inegalitatea $x^3+y^3 \geq xy(x+y) \Leftrightarrow (x+y)(x-y)^2 \geq 0$, $x,y>0$ obținem:

$$Ms = \sum \frac{1}{b^3+c^3+\lambda abc} \leq \sum \frac{1}{bc(b+c)+\lambda abc} = \sum \frac{1}{bc(b+c+\lambda a)} =$$

$$= \frac{1}{abc}\sum \frac{a}{b+c+\lambda a} \overset{(1)}{\leq} \frac{1}{abc}\cdot\frac{3}{\lambda+2} = \frac{3}{(\lambda+2)abc} = Md,$$

unde (1) $\Leftrightarrow \sum \frac{a}{b+c+\lambda a} \leq \frac{3}{\lambda+2} \overset{reverse}{\Leftrightarrow} \sum \frac{b+c}{b+c+\lambda a} \geq \frac{6}{\lambda+2}$.

Obținem:

$$\sum \frac{b+c}{b+c+\lambda a} = \sum \frac{(b+c)^2}{(b+c)^2+\lambda a(b+c)} \overset{CS}{\geq} \frac{\left[\sum(b+c)\right]^2}{\sum\left[(b+c)^2+\lambda a(b+c)\right]} = \frac{4\left(\sum a\right)^2}{2\sum a^2+2(\lambda+1)\sum bc} =$$

$\sum a^2 \geq \sum bc$ și $(\lambda-1) \geq 0$, adevărată din condiția din ipoteză $\lambda \geq 1$.

Egalitatea are loc dacă și numai dacă $a=b=c$.

Problema132.

If $a,b,c>0$ then

$$\frac{ab+2a+b}{a+2b+1}+\frac{ac+2a+c}{a+2c+1} \geq \frac{4a}{a+1}.$$

Daniel Sitaru, Romania

Soluție.

Lema.

If $a,b,c>0$ then

$$\frac{ab+2a+b}{a+2b+1} \geq \frac{2a}{a+1}.$$

Demonstrație.

$\frac{ab+2a+b}{a+2b+1} \geq \frac{2a}{a+1} \Leftrightarrow b(a-1)^2 \geq 0$, cu egalitate pentru $a=1$.

$LHS = \frac{ab+2a+b}{a+2b+1} + \frac{ac+2a+c}{a+2c+1} \overset{Lema}{\geq} \frac{2a}{a+1} + \frac{2a}{a+1} = \frac{4a}{a+1} = RHS$.

Egalitatea are loc daca si numai daca $a=1$.

Remarca.

If $a_1, a_2, ..., a_n, \lambda > 0$ then

$$\frac{\lambda a_1 + 2\lambda + a_1}{\lambda + 2a_1 + 1} + \frac{\lambda a_2 + 2\lambda + a_2}{\lambda + 2a_2 + 1} + ... + \frac{\lambda a_n + 2\lambda + a_n}{\lambda + 2a_n + 1} \geq \frac{2n\lambda}{\lambda + 1}.$$

Marin Chirciu

Soluție.

Lema.

If $\lambda, a_1 > 0$ then

$$\frac{\lambda a_1 + 2\lambda + a_1}{\lambda + 2a_1 + 1} \geq \frac{2\lambda}{\lambda + 1}.$$

Demonstrație.

$\frac{\lambda a_1 + 2\lambda + a_1}{\lambda + 2a_1 + 1} \geq \frac{2\lambda}{\lambda+1} \Leftrightarrow a_1(\lambda-1)^2 \geq 0$, cu egalitate pentru $\lambda = 1$.

$LHS = \sum \frac{\lambda a_1 + 2\lambda + a_1}{\lambda + 2a_1 + 1} \overset{Lema}{\geq} \sum \frac{2\lambda}{\lambda+1} = \frac{2n\lambda}{\lambda+1} = RHS$.

Egalitatea are loc daca si numai daca $\lambda = 1$.

Problema133.

In ΔABC

$$\frac{9r}{2}\sqrt[3]{\frac{r^2}{2R^2}} \leq \sum h_a \sin^2\frac{A}{2} \leq \frac{9R}{8}.$$

Mathematical Inequalities 5/2024, George Apostolopoulos, Greece

Soluție.

Lema.

In ΔABC

$$\sum h_a \sin^2\frac{A}{2} = \frac{r(4R+r)}{2R}.$$

Inegalitatea din dreapta.

$\sum h_a \sin^2\frac{A}{2} \leq \frac{9R}{8} \Leftrightarrow \frac{r(4R+r)}{2R} \leq \frac{9R}{8} \Leftrightarrow 9R^2 - 16Rr - 4r^2 \geq 0 \Leftrightarrow (R-2r)(9R+2r) \geq 0$,

vezi $R \geq 2r$, (Euler).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Inegalitatea din stanga:

$\sum h_a \sin^2\frac{A}{2} \geq \frac{9r}{2}\sqrt[3]{\frac{r^2}{2R^2}} \Leftrightarrow \frac{r(4R+r)}{2R} \geq \frac{9r}{2}\sqrt[3]{\frac{r^2}{2R^2}} \Leftrightarrow \frac{(4R+r)}{R} \geq 9\sqrt[3]{\frac{r^2}{2R^2}} \Leftrightarrow$

$\frac{(4R+r)^3}{R^3} \geq 729 \cdot \frac{r^2}{2R^2} \Leftrightarrow 2(4R+r)^3 \geq 729Rr^2 \Leftrightarrow 128R^3 + 96R^2r - 705Rr^2 + 2r^3 \geq 0 \Leftrightarrow$

$\Leftrightarrow (R-2r)(128R^2 + 352Rr - r^2) \geq 0$, vezi $R \geq 2r$, (Euler).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\frac{9R}{8}\le\sum r_a\sin^2\frac{A}{2}\le\frac{9R^3}{32r^2}.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\sum r_a\sin^2\frac{A}{2}=\frac{8R^2+2Rr-p^2}{2R}$$

Inegalitatea din dreapta.

$$\sum r_a\sin^2\frac{A}{2}=\frac{8R^2+2Rr-p^2}{2R}\overset{Gerretsen}{\le}\frac{8R^2+2Rr-16Rr+5r^2}{2R}=\frac{8R^2-14Rr+5r^2}{2R}\overset{Euler}{\le}\frac{9R^3}{32r^2}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Inegalitatea din stanga:

$$\sum r_a\sin^2\frac{A}{2}=\frac{8R^2+2Rr-p^2}{2R}\overset{Gerretsen}{\ge}\frac{8R^2+2Rr-4R^2-4Rr-3r^2}{2R}=\frac{4R^2-2Rr-3r^2}{2R}\overset{Euler}{\ge}\frac{9R}{8}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum h_a\sin^2\frac{A}{2}\le\frac{9R}{8}\le\sum r_a\sin^2\frac{A}{2}.$$

Marin Chirciu

Soluție.

Folosind **Lemele** de mai sus, dubla inegalitate se scrie:

$$\frac{r(4R+r)}{2R}\le\frac{9R}{8}\le\frac{8R^2+2Rr-p^2}{2R}.$$

Prima inegalitate $\frac{r(4R+r)}{2R}\le\frac{9R}{8}\Leftrightarrow 9R^2-16Rr-4r^2\ge 0\Leftrightarrow (R-2r)(9R+2r)\ge 0$,

vezi $R\ge 2r$, (Euler).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

A doua inegalitate $\frac{9R}{8}\le\frac{8R^2+2Rr-p^2}{2R}\Leftrightarrow 4p^2\le 23R^2+8Rr$, vezi $p^2\le 4R^2+4Rr+3r^2$.

Rămâne să arătăm că:

$$4\left(4R^2+4Rr+3r^2\right)\le 23R^2+8Rr\Leftrightarrow 7R^2-8Rr-12r^2\ge 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(7R+6r)\ge 0$, vezi $R\ge 2r$, (Euler).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$1).\ \frac{27r^2}{2R}\le\sum h_a\cos^2\frac{A}{2}\le\frac{27R}{8}.$$

Soluție.

Lema.

In ΔABC

$$\sum r_a\cos^2\frac{A}{2}=\frac{p^2}{2R}$$

$$2).\ \sum h_a\cos^2\frac{A}{2}=\sum r_a\cos^2\frac{A}{2}.$$

$$3).\ 3\sum h_a \sin^2\frac{A}{2} \le \sum h_a \cos^2\frac{A}{2}.$$

$$4).\ 3\sum r_a \sin^2\frac{A}{2} \ge \sum r_a \cos^2\frac{A}{2}.$$

Dezvoltări, Marin Chirciu

Problema134.

If $a,b\in\mathbf{R}, (a+b)^5+4ab\ge 36$ then

$$a^3+b^3+a+b\ge 4.$$

RMM 5/2024, Nguyen Hung Cuong, Vietnam

Soluție.

Lema.

If $a,b\in\mathbf{R}, (a+b)^5+4ab\ge 36$ then

$$a+b\ge 2.$$

Demonstrație.

$$36\le (a+b)^5+4ab\le (a+b)^5+(a+b)^2 \Rightarrow 36\le (a+b)^5+(a+b)^2 \Rightarrow 36\le t^5+t^2 \Leftrightarrow$$

$$\Leftrightarrow t^5+t^2-36\ge 0 \Leftrightarrow (t-2)(t^4+2t^3+4t^2+9t+18)\ge 0 \Rightarrow t\ge 2 \Rightarrow a+b\ge 2.$$

$$LHS=a^3+b^3+a+b \overset{Holder}{\ge} \frac{(a+b)^3}{4}+a+b \overset{Lema}{\ge} \frac{2^3}{4}+2=4=RHS$$

Egalitatea are loc daca si numai daca $a=b=1$.

Remarca.

If $a,b\in\mathbf{R}, (a+b)^6+4ab\ge 68$ then

$$a^5+b^5+a^3+b^3+a+b\ge 6.$$

Marin Chirciu

Soluție.

Lema.

If $a,b\in\mathbf{R}, (a+b)^6+4ab\ge 68$ then

$$a+b\ge 2.$$

Demonstrație.

$$68\le (a+b)^6+4ab\le (a+b)^6+(a+b)^2 \Rightarrow 68\le (a+b)^6+(a+b)^2 \Rightarrow 68\le t^6+t^2 \Leftrightarrow$$

$$\Leftrightarrow t^6+t^2-68\ge 0 \Leftrightarrow (t-2)(t^5+2t^4+4t^3+8t^2+17t+34)\ge 0 \Rightarrow t\ge 2 \Rightarrow a+b\ge 2.$$

$$LHS=a^5+b^5+a^3+b^3+a+b \overset{Holder}{\ge} \frac{(a+b)^5}{16}+\frac{(a+b)^3}{4}+a+b \overset{Lema}{\ge} \frac{2^5}{16}+\frac{2^3}{4}+2=6=RHS$$

Egalitatea are loc daca si numai daca $a=b=1$.

Problema135.

In ΔABC

$$6\le \sum \frac{\cos\frac{B-C}{2}}{\cos\frac{B+C}{2}} \le \frac{3R}{r}.$$

Mathematical Inequalities 5/2024, George Apostolopoulos, Greece

Soluție

Inegalitatea din dreapta:

$$\sum \frac{\cos\frac{B-C}{2}}{\cos\frac{B+C}{2}} \overset{\cos x\le 1}{\le} \sum \frac{1}{\sin\frac{A}{2}} \overset{(1)}{\le} \frac{3R}{r},$$

unde(1) $\Leftrightarrow \sum \frac{1}{\sin\frac{A}{2}} \overset{(1)}{\le} \frac{3R}{r}$ care rezultă din: $\sum \frac{1}{\sin\frac{A}{2}} \le 2\left(1+\frac{R}{r}\right) \overset{Euler}{\le} \frac{3R}{r}$.

Inegalitatea din stanga:

$$\sum \frac{\cos\frac{B-C}{2}}{\cos\frac{B+C}{2}} \overset{AM-GM}{\ge} 3\sqrt[3]{\prod \frac{\cos\frac{B-C}{2}}{\cos\frac{B+C}{2}}} = 3\sqrt[3]{\frac{\prod\cos\frac{B-C}{2}}{\prod\sin\frac{A}{2}}} = 3\sqrt[3]{\frac{\frac{p^2+r^2+2Rr}{8R^2}}{\frac{r}{4R}}} =$$

$$= 3\sqrt[3]{\frac{p^2+r^2+2Rr}{2Rr}} \overset{Gerrtsen}{\ge} 3\sqrt[3]{\frac{16Rr-5r^2+r^2+2Rr}{2Rr}} = 3\sqrt[3]{\frac{18Rr-4r^2}{2Rr}} = 3\sqrt[3]{\frac{9R-2r}{R}} \overset{Euler}{\ge} 6.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$3\sqrt[3]{9-\frac{2r}{R}} \le \sum \frac{\cos\frac{B-C}{2}}{\cos\frac{B+C}{2}} \le 2\left(1+\frac{R}{r}\right).$$

Marin Chirciu

Remarca.

In ΔABC

$$6 \le 3\sqrt[3]{9-\frac{2r}{R}} \le \sum \frac{\cos\frac{B-C}{2}}{\cos\frac{B+C}{2}} \le 2\left(1+\frac{R}{r}\right) \le \frac{3R}{r}.$$

Problema136.

In ΔABC

$$\sum \frac{a}{b+c}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right) \ge 3\sqrt{3}.$$

RMM 5/2024, Zaza Mzhavanadze, Georgia

Soluție.

Lema.

In ΔABC

$$\sum \frac{a}{b+c}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right) = \frac{2p\left(p^2-3r^2-4Rr\right)}{r\left(p^2+r^2+2Rr\right)}.$$

$$LHS = \sum \frac{a}{b+c}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right) = \frac{2p\left(p^2-3r^2-4Rr\right)}{r\left(p^2+r^2+2Rr\right)} \overset{(1)}{\ge} 3\sqrt{3} = RHS,$$

unde(1) $\Leftrightarrow \frac{2p\left(p^2-3r^2-4Rr\right)}{r\left(p^2+r^2+2Rr\right)} \ge 3\sqrt{3}$, care rezultă din inegalitatea lui Mitrinovic $p \ge 3\sqrt{3}r$.

Rămâne să arătăm că:

$$\frac{2\cdot 3\sqrt{3}r\left(p^2-3r^2-4Rr\right)}{r\left(p^2+r^2+2Rr\right)}\geq 3\sqrt{3}\Leftrightarrow 2\left(p^2-3r^2-4Rr\right)\geq p^2+r^2+2Rr\Leftrightarrow$$

$\Leftrightarrow p^2\geq 10Rr+7r^2$**,** care rezultă din inegalitatea lui Gerretsen $p^2\geq 16Rr-5r^2$.
Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sum\frac{a}{b+c}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)\leq 3\sqrt{3}\left(\frac{R}{2r}\right)^2.$$

Marin Chirciu

Soluție.

Folosind **Lema**de mai sus, obținem:

$$LHS=\sum\frac{a}{b+c}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)=\frac{2p\left(p^2-3r^2-4Rr\right)}{r\left(p^2+r^2+2Rr\right)}\overset{(1)}{\leq}3\sqrt{3}\left(\frac{R}{2r}\right)^2=RHS,$$

unde(1) $\Leftrightarrow\frac{2p\left(p^2-3r^2-4Rr\right)}{r\left(p^2+r^2+2Rr\right)}\overset{(1)}{\leq}3\sqrt{3}\left(\frac{R}{2r}\right)^2$, vezi inegalitatea lui Mitrinovic $p\leq\frac{3\sqrt{3}R}{2}$.

Rămâne să arătăm că:

$$\frac{3\sqrt{3}R\left(p^2-3r^2-4Rr\right)}{r\left(p^2+r^2+2Rr\right)}\leq 3\sqrt{3}\frac{R^2}{4r^2}\Leftrightarrow\frac{\left(p^2-3r^2-4Rr\right)}{\left(p^2+r^2+2Rr\right)}\leq\frac{R}{4r}\Leftrightarrow$$

$$4r\left(p^2-3r^2-4Rr\right)\leq R\left(p^2+r^2+2Rr\right)\Leftrightarrow p^2\left(R-4r\right)+r\left(2R^2+17Rr+12r^2\right)\geq 0.$$

Distingem cazurile:

Cazul1). Dacă $\left(R-4r\right)\geq 0$ inegalitatea este evidentă.

Cazul2). Dacă $\left(R-4r\right)<0$ inegalitatea se rescrie: $r\left(2R^2+17Rr+12r^2\right)\geq p^2\left(4r-R\right)$, care rezultă din inegalitatea lui Gerretsen: $p^2\leq 4R^2+4Rr+3r^2$.

Rămâne să arătăm că:

$$r\left(2R^2+17Rr+12r^2\right)\geq\left(\left(4R^2+4Rr+3r^2\right)\right)\left(4r-R\right)\Leftrightarrow 4R^3-10R^2r+4Rr^2\geq 0\Leftrightarrow$$

$\Leftrightarrow R\left(R-2r\right)\left(2R-r\right)\geq 0$, vezi $R\geq 2r$,(Euler).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\sqrt{3}\cdot\frac{2r}{R}\sum\frac{a}{b+c}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\leq\sqrt{3}\cdot\frac{R}{2r}.$$

Marin Chirciu

Soluție.

Lema.

In ΔABC

$$\sum\frac{a}{b+c}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\frac{2\left[p^2\left(2R+3r\right)-r\left(8R^2+6Rr+r^2\right)\right]}{p\left(p^2+r^2+2Rr\right)}.$$

Inegalitatea din dreapta.

$$\sum\frac{a}{b+c}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\frac{2\left[p^2\left(2R+3r\right)-r\left(8R^2+6Rr+r^2\right)\right]}{p\left(p^2+r^2+2Rr\right)}\overset{(1)}{\leq}\sqrt{3}\cdot\frac{R}{2r},$$

unde (1) $\Leftrightarrow \dfrac{2\left[p^2(2R+3r)-r\left(8R^2+6Rr+r^2\right)\right]}{p\left(p^2+r^2+2Rr\right)} \le \sqrt{3}\cdot\dfrac{R}{2r} \Leftrightarrow$

$\Leftrightarrow 4r\left[p^2(2R+3r)-r\left(8R^2+6Rr+r^2\right)\right] \le \sqrt{3}\cdot Rp\left(p^2+r^2+2Rr\right)$, care rezultă din inegalitatea lui Mitrinovic $p \ge 3\sqrt{3}r$.

Rămâne să arătăm că:

$4r\left[p^2(2R+3r)-r\left(8R^2+6Rr+r^2\right)\right] \le \sqrt{3}\cdot R\cdot 3\sqrt{3}r\left(p^2+r^2+2Rr\right) \Leftrightarrow$

$\Leftrightarrow 4\left[p^2(2R+3r)-r\left(8R^2+6Rr+r^2\right)\right] \le 9R\left(p^2+r^2+2Rr\right) \Leftrightarrow$

$\Leftrightarrow p^2(R-12r)+r\left(40R^2+33Rr+4r^2\right) \ge 0$.

Distingem cazurile:

Cazul1). Dacă $(R-12r)\ge 0$ inegalitatea este evidentă.

Cazul2). Dacă $(R-12r)<0$ inegalitatea se rescrie: $r\left(40R^2+33Rr+4r^2\right) \ge p^2(12r-R)$, care rezultă din inegalitatea lui Gerretsen: $p^2 \le 4R^2+4Rr+3r^2$.

Rămâne să arătăm că:

$r\left(40R^2+33Rr+4r^2\right) \ge \left(4R^2+4Rr+3r^2\right)(12r-R) \Leftrightarrow 2R^3+3R^2r-6Rr^2-16r^3 \ge 0 \Leftrightarrow$

$\Leftrightarrow (R-2r)\left(2R^2+7Rr+8r^2\right)\ge 0$, vezi $R\ge 2r$,(Euler).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Inegalitatea din stânga.

$$\sum\frac{a}{b+c}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=\frac{2\left[p^2(2R+3r)-r\left(8R^2+6Rr+r^2\right)\right]}{p\left(p^2+r^2+2Rr\right)} \overset{(2)}{\ge} \sqrt{3}\cdot\frac{2r}{R},$$

unde (2) $\Leftrightarrow \dfrac{2\left[p^2(2R+3r)-r\left(8R^2+6Rr+r^2\right)\right]}{p\left(p^2+r^2+2Rr\right)} \ge \sqrt{3}\cdot\dfrac{2r}{R} \Leftrightarrow$

$\Leftrightarrow 2R\left[p^2(2R+3r)-r\left(8R^2+6Rr+r^2\right)\right] \ge \sqrt{3}\cdot 2rp\left(p^2+r^2+2Rr\right)$, care rezultă din inegalitatea lui Mitrinovic $p \le \dfrac{3\sqrt{3}R}{2}$.

Rămâne să arătăm că:

$2R\left[p^2(2R+3r)-r\left(8R^2+6Rr+r^2\right)\right] \ge \sqrt{3}\cdot 2r\cdot\dfrac{3\sqrt{3}R}{2}\left(p^2+r^2+2Rr\right) \Leftrightarrow$

$\Leftrightarrow 2\left[p^2(2R+3r)-r\left(8R^2+6Rr+r^2\right)\right] \ge 9r\left(p^2+r^2+2Rr\right) \Leftrightarrow$

$\Leftrightarrow p^2(4R-3r)+r\left(16R^2+30Rr+11r^2\right)\ge 0$,vezi inegalitatea lui Gerretsen: $p^2 \le 16Rr-5r^2$.

Rămâne să arătăm că:

$\left(16Rr-5r^2\right)(4R-3r)+r\left(16R^2+30Rr+11r^2\right)\ge 0 \Leftrightarrow 24R^2-49Rr^2+2r^2 \ge 0 \Leftrightarrow$

$\Leftrightarrow (R-2r)(24R-r)\ge 0$, vezi $R\ge 2r$,(Euler).

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$3\sum\frac{a}{b+c}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)\le\sum\frac{a}{b+c}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right).$$

Marin Chirciu

Soluție.
Folosind **Lemele** de mai sus, inegalitatea se scrie:

$$3\cdot\frac{2\left[p^2(2R+3r)-r\left(8R^2+6Rr+r^2\right)\right]}{p\left(p^2+r^2+2Rr\right)}\le\frac{2p\left(p^2-3r^2-4Rr\right)}{r\left(p^2+r^2+2Rr\right)}\Leftrightarrow$$

$$\Leftrightarrow 3r\left[p^2(2R+3r)-r\left(8R^2+6Rr+r^2\right)\right]\le p^2\left(p^2-3r^2-4Rr\right)\Leftrightarrow$$

$$\Leftrightarrow p^2\left(p^2-12r^2-10Rr\right)+3r^2\left(8R^2+6Rr+r^2\right)\ge 0.$$

Distingem cazurile:
Cazul1). Dacă $\left(p^2-12r^2-10Rr\right)\ge 0$ inegalitatea este evidentă.
Cazul2). Dacă $\left(p^2-12r^2-10Rr\right)<0$ inegalitatea se rescrie:

$$3r^2\left(8R^2+6Rr+r^2\right)\ge p^2\left(10Rr+12r^2-p^2\right),$$

care rezultă din inegalitatea lui Gerretsen: $16Rr-5r^2\le p^2\le 4R^2+4Rr+3r^2$.
Rămâne să arătăm că:

$$3r^2\left(8R^2+6Rr+r^2\right)\ge\left(4R^2+4Rr+3r^2\right)\left(10Rr+12r^2-16Rr+5r^2\right)\Leftrightarrow$$

$$3r\left(8R^2+6Rr+r^2\right)\ge\left(4R^2+4Rr+3r^2\right)(-6R+17r)\Leftrightarrow 6R^3-5R^2r-8Rr^2-12r^3\ge 0$$

$$\Leftrightarrow (R-2r)\left(6R^2+7Rr+6r^2\right)\ge 0,\text{ vezi } R\ge 2r,\text{(Euler).}$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Problema137.
In ΔABC

$$\sum a\sqrt{4a^2+9b^2}\ge 10\sqrt{6}F.$$

RMM 5/2024, Daniel Sitaru, Romania

Soluție.
Lema.
If $a,b>0$ then

$$4a^2+9b^2\overset{CS}{\ge}\frac{(2a+3b)^2}{2}.$$

Demonstrație.

$$4a^2+9b^2=\frac{(2a)^2}{1}+\frac{(3b)^2}{1}\overset{CS}{\ge}\frac{(2a+3b)^2}{1+1},\text{ cu egalitate pentru }\frac{2a}{1}=\frac{3b}{1}.$$

$$LHS=\sum a\sqrt{4a^2+9b^2}\overset{CS}{\ge}\sum a\sqrt{\frac{(2a+3b)^2}{2}}=\frac{1}{\sqrt{2}}\sum a(2a+3b)=\frac{2\sum a^2+3\sum ab}{\sqrt{2}}\overset{SOS}{\ge}$$

$$\overset{SOS}{\ge}\frac{2\sum ab+3\sum ab}{\sqrt{2}}=\frac{5\sum ab}{\sqrt{2}}\overset{Gordon}{\ge}\frac{5\cdot 4\sqrt{3}F}{\sqrt{2}}=10\sqrt{6}F=RHS.$$

Am folosit mai sus $\sum ab\ge 4\sqrt{3}F$.

Egalitatea are loc daca si numai daca $\frac{a}{b}=\frac{b}{c}=\frac{c}{a}=\frac{3}{2}$, contradicție.

Inegalitatea este strictă.

Remarca.
In ΔABC

$$\sum a\sqrt{\lambda^2a^2+n^2b^2}\ge 10\sqrt{6}F\text{ ,unde }\lambda,n>0.$$

Marin Chirciu

Soluție.
Lema.
If $a,b,\lambda,n>0$ then

$$\lambda^2a^2+n^2b^2\overset{CS}{\geq}\frac{(\lambda a+nb)^2}{2}.$$

Demonstrație.
$\lambda^2a^2+n^2b^2=\frac{(\lambda a)^2}{1}+\frac{(nb)^2}{1}\overset{CS}{\geq}\frac{(\lambda a+nb)^2}{1+1}$, cu egalitate pentru $\frac{\lambda a}{1}=\frac{nb}{1}$.
Să trecem la rezolvarea problemei din enunț.

$$LHS=\sum a\sqrt{\lambda^2a^2+n^2b^2}\overset{CS}{\geq}\sum a\sqrt{\frac{(\lambda a+nb)^2}{2}}=\frac{1}{\sqrt{2}}\sum a(\lambda a+nb)=\frac{\lambda\sum a^2+n\sum ab}{\sqrt{2}}\overset{SOS}{\geq}$$

$$\overset{SOS}{\geq}\frac{\lambda\sum ab+n\sum ab}{\sqrt{2}}=\frac{(\lambda+n)\sum ab}{\sqrt{2}}\overset{Gordon}{\geq}\frac{(\lambda+n)\cdot 4\sqrt{3}F}{\sqrt{2}}=2(\lambda+n)\sqrt{6}F=RHS.$$

Egalitatea are loc daca si numai daca $\frac{a}{b}=\frac{b}{c}=\frac{c}{a}=\frac{n}{\lambda}\Leftrightarrow\lambda=n$.

Am folosit mai sus $\sum ab\geq 4\sqrt{3}F$.

Egalitatea are loc dacă și numai dacă $\lambda=n$ și triunghiul este echilateral.

Problema138.
L:1118. Calculați

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos 2x}{e^x+1}dx.$$

Sclipirea Minții 33/2024,Vasile Mircea Popa, Sibiu

Soluție.

$$I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos 2x}{e^x+1}dx\overset{x\to -x}{=}\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}}\frac{\cos(-2x)}{e^{-x}+1}(-dx)=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos 2x}{\frac{1}{e^x}+1}dx=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{e^x\cos 2x}{e^x+1}dx.$$

Adunînd $I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos 2x}{e^x+1}dx$ și $I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{e^x\cos 2x}{e^x+1}dx$ obținem:

$$2I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos 2x}{e^x+1}dx+\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{e^x\cos 2x}{e^x+1}dx=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{(e^x+1)\cos 2x}{e^x+1}dx=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos 2x dx=\frac{\sin 2x}{2}\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=$$

$$=\frac{1}{2}\left(\sin\frac{\pi}{2}-\sin\frac{-\pi}{2}\right)=\frac{1}{2}(1+1)=1.$$

Din $2I=1\Rightarrow I=\frac{1}{2}$.

În final deducem că $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos 2x}{e^x+1}dx=\frac{1}{2}$.

Remarca.

Calculați

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos nx}{e^x+1}dx \text{ ,unde } n \in \mathbf{N}.$$

Marin Chirciu

Soluție.

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos nx}{e^x+1}dx \overset{x\to -x}{=} \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \frac{\cos(-nx)}{e^{-x}+1}(-dx) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos nx}{\frac{1}{e^x}+1}dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{e^x\cos nx}{e^x+1}dx.$$

Adunînd $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos nx}{e^x+1}dx$ și $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{e^x\cos nx}{e^x+1}dx$ obținem:

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos nx}{e^x+1}dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{e^x\cos nx}{e^x+1}dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(e^x+1)\cos nx}{e^x+1}dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos nx dx = \frac{\sin nx}{n}\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} =$$

$$= \frac{1}{n}\left(\sin\frac{n\pi}{4} - \sin\frac{-n\pi}{4}\right) = \frac{1}{n}\cdot 2\sin\frac{n\pi}{4} = \frac{2}{n}\sin\frac{n\pi}{4}, \text{ pentru } n>0.$$

Din $2I = \frac{2}{n}\sin\frac{n\pi}{4} \Rightarrow I = \frac{1}{n}\sin\frac{n\pi}{4}$.

Pentru $n=0 \Rightarrow 2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dx = x\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \left(\frac{\pi}{4} - \frac{-\pi}{4}\right) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$.

În final deducem că $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos nx}{e^x+1}dx = \frac{1}{n}\sin\frac{n\pi}{4}$, pentru $n>0$ și $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{e^x+1}dx = \frac{\pi}{4}$.

Remarca.
Calculați

$$\lim_{n\to\infty} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos nx}{e^x+1}dx.$$

Marin Chirciu

Soluție.

$$I_n = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos nx}{e^x+1}dx \overset{x\to -x}{=} \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \frac{\cos(-nx)}{e^{-x}+1}(-dx) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos nx}{\frac{1}{e^x}+1}dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{e^x\cos nx}{e^x+1}dx.$$

Adunînd $I_n = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos nx}{e^x+1}dx$ și $I_n = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{e^x\cos nx}{e^x+1}dx$ obținem:

$$2I_n=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos nx}{e^x+1}dx+\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{e^x\cos nx}{e^x+1}dx=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{(e^x+1)\cos nx}{e^x+1}dx=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos nxdx=\frac{\sin nx}{n}\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=$$

$$=\frac{1}{n}\left(\sin\frac{n\pi}{4}-\sin\frac{-n\pi}{4}\right)=\frac{1}{n}\cdot 2\sin\frac{n\pi}{4}=\frac{2}{n}\sin\frac{n\pi}{4}.$$

Din $2I_n=\frac{2}{n}\sin\frac{n\pi}{4}\Rightarrow I_n=\frac{1}{n}\sin\frac{n\pi}{4}$.

Calculăm $\lim\limits_{n\to\infty}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos nx}{e^x+1}dx=\lim\limits_{n\to\infty}\frac{1}{n}\sin\frac{n\pi}{4}=0$.

În final deducem că $\lim\limits_{n\to\infty}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos nx}{e^x+1}dx=0$.

Problema139.

S:L24.54 If $a,b,c>1$ then

$$\frac{\log_a^4 b}{a}+\frac{\log_b^4 c}{b}+\frac{\log_c^4 a}{c}\geq\frac{9}{a+b+c}.$$

SGM 2/2024, Liliana Tomiță, Suceava

Soluție.

$$LHS=\frac{\log_a^4 b}{a}+\frac{\log_b^4 c}{b}+\frac{\log_c^4 a}{c}\overset{Holder}{\geq}\frac{(\log_a b+\log_b c+\log_c a)^4}{3^2(a+b+c)}\overset{AM-GM}{\geq}\frac{\left(3\sqrt[3]{\log_a b\cdot\log_b c\cdot\log_c a}\right)^4}{3^2(a+b+c)}=$$

$$=\frac{(3\cdot 1)^4}{3^2(a+b+c)}=\frac{9}{a+b+c}=RHS.$$

Egalitatea are loc daca si numai daca $a=b=c$.

Remarca.

If $a,b,c>1$ and $n\in\mathbf{N}$ then

$$\frac{\log_a^n b}{a}+\frac{\log_b^n c}{b}+\frac{\log_c^n a}{c}\geq\frac{9}{a+b+c}.$$

Marin Chirciu

Problema140

28802.Calculati

$$\lim_{n\to\infty}\int_{\frac{1}{n}}^{n}\frac{1}{(x^2+1)(x^a+1)}dx,\text{ unde } a\in\mathbf{R}.$$

GM 2/2024,Anișoara Rotariu, Dorohoi

Soluție.

$$I_n=\int_{\frac{1}{n}}^{n}\frac{1}{(x^2+1)(x^a+1)}dx\overset{x\to\frac{1}{x}}{=}\int_{n}^{\frac{1}{n}}\frac{1}{\left(\frac{1}{x^2}+1\right)\left(\frac{1}{x^a}+1\right)}\left(\frac{-1}{x^2}\right)dx=\int_{\frac{1}{n}}^{n}\frac{x^a}{(x^2+1)(x^a+1)}dx.$$

Obtinem:

$$2I_n=I_n+I_n=\int_{\frac{1}{n}}^{n}\frac{1}{(x^2+1)(x^a+1)}dx+\int_{\frac{1}{n}}^{n}\frac{x^a}{(x^2+1)(x^a+1)}dx=\int_{\frac{1}{n}}^{n}\frac{x^a+1}{(x^2+1)(x^a+1)}dx=$$

$$=\int_{\frac{1}{n}}^{n}\frac{1}{x^2+1}dx=arctgx\Big|_{\frac{1}{n}}^{n}=arctgn-arctg\frac{1}{n}.$$

Calculam $\lim\limits_{n\to\infty}\left(arctgn-arctg\frac{1}{n}\right)=\frac{\pi}{2}-0=\frac{\pi}{2}$.

Deducem ca $\lim\limits_{n\to\infty}\int_{\frac{1}{n}}^{n}\frac{1}{(x^2+1)(x^a+1)}dx=\frac{\pi}{4}$.

Problema141.

Q104. If $(a_n)_{n\geq1}$ is a real positive sequence such that $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=e$,then compute

$$\lim_{n\to\infty}\sqrt{n}\left(\sqrt{a_{n+1}}-\sqrt{a_n}\right).$$

Sclipirea Minții Nr33, D.M.Batinetu-Giurgiu, Bucharest, Romania

Solution

Lemma.

If $(a_n)_{n\geq1}$ is a real positive sequence such that $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=\lambda>0$,then $\lim\limits_{n\to\infty}a_n=\infty$.

$$\lim_{n\to\infty}\sqrt{n}\left(\sqrt{a_{n+1}}-\sqrt{a_n}\right)=\lim_{n\to\infty}\frac{\sqrt{n}(a_{n+1}-a_n)}{\sqrt{a_{n+1}}+\sqrt{a_n}}=\lim_{n\to\infty}\frac{a_{n+1}-a_n}{\sqrt{\frac{a_{n+1}}{n}}+\sqrt{\frac{a_n}{n}}}=\frac{\lim\limits_{n\to\infty}(a_{n+1}-a_n)}{\lim\limits_{n\to\infty}\sqrt{\frac{a_{n+1}}{n}}+\lim\limits_{n\to\infty}\sqrt{\frac{a_n}{n}}}=$$

$$=\frac{e}{\sqrt{e}+\sqrt{e}}=\frac{\sqrt{e}}{2}$$,which results from: $\lim\limits_{n\to\infty}\frac{a_n}{n}\overset{\text{Stolz-Cesaro}}{=}\lim\limits_{n\to\infty}\frac{a_{n+1}-a_n}{n+1-n}=e$

and $\lim\limits_{n\to\infty}\frac{a_{n+1}}{n}=\lim\limits_{n\to\infty}\left(\frac{a_{n+1}}{n+1}\cdot\frac{n+1}{n}\right)=\lim\limits_{n\to\infty}\frac{a_{n+1}}{n+1}\cdot\lim\limits_{n\to\infty}\frac{n+1}{n}=e\cdot1=e$.

Deduce $\lim\limits_{n\to\infty}\sqrt{n}\left(\sqrt{a_{n+1}}-\sqrt{a_n}\right)=\frac{\sqrt{e}}{2}$.

Remark.

If $(a_n)_{n\geq1}$ is a real positive sequence such that $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=\lambda>0$,then compute

$$\lim_{n\to\infty}\sqrt{n}\left(\sqrt{a_{n+1}}-\sqrt{a_n}\right).$$

Solution

Deduce $\lim\limits_{n\to\infty}\sqrt{n}\left(\sqrt{a_{n+1}}-\sqrt{a_n}\right)=\frac{\sqrt{\lambda}}{2}$.

Problema142.

Q103. Compute

$$\lim_{n\to\infty}\frac{2\left[\sqrt[3]{n^3+3n^2+1}\right]-\left[\sqrt{n^2+2n}\right]}{\sqrt[n]{n!}}.$$

Sclipirea Minții Nr33/2024, Florin Rotaru, Focsani, Romania

Solution

Using $x-1<[x]\leq x$, $\forall x\in\mathbf{R}$ we have:

$$\frac{2\sqrt[3]{n^3+3n^2+1}-\sqrt{n^2+2n}-2}{\sqrt[n]{n!}}<\frac{2\left[\sqrt[3]{n^3+3n^2+1}\right]-\left[\sqrt{n^2+2n}\right]}{\sqrt[n]{n!}}<\frac{2\sqrt[3]{n^3+3n^2+1}-\sqrt{n^2+2n}-1}{\sqrt[n]{n!}}$$

Cu teorema clestelui obtinem:

$$\lim_{n\to\infty}\frac{2\left[\sqrt[3]{n^3+3n^2+1}\right]-\left[\sqrt{n^2+2n}\right]}{\sqrt[n]{n!}}=e, \text{ deoarece:}$$

$$\lim_{n\to\infty}\frac{\sqrt[3]{n^3+3n^2+1}}{\sqrt[n]{n!}}=e, \text{ vezi:}$$

$$\lim_{n\to\infty}\frac{\sqrt[3]{n^3+3n^2+1}}{\sqrt[n]{n!}}=\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\sqrt[3]{1+\frac{3}{n}+\frac{1}{n^2}}=\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}}=\lim_{n\to\infty}\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}=$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e.$$

Analog $\lim_{n\to\infty}\frac{\sqrt{n^2+2n}}{\sqrt[n]{n!}}=e$ si $\lim_{n\to\infty}\frac{1}{\sqrt[n]{n!}}=0$.

Problema143.

Fie functia $f:\mathbf{R}\to\mathbf{R}$,cu $f(0)=-2$, care admite primitiva F.

Daca $f(x)+2F(x)=3, \forall x\in\mathbf{R}$ sa se calculeze $f(-1)$.

Pre-Admitere Universitatea Poltehnica 2024

Soluție.

Avem $F'(x)=f(x)$.

Se deriveaza $f(x)+2F(x)=3 \Rightarrow f'(x)+2F'(x)=0 \Rightarrow f'(x)+2f(x)=0$.

Ecuatia diferentiala omogena are solutia generala de forma $f(x)=Ce^{-2x}$, unde C este constanta care se determina din conditia initiala $f(0)=-2 \Rightarrow f(0)=Ce^{-2\cdot 0}=C \Rightarrow C=-2$.

O solutie particulara a ecuatiei diferentiale este $f(x)=-2e^{-2x} \Rightarrow f(-1)=-2e^{-2\cdot(-1)}=-2e^2$.

Deducem ca $f(-1)=-2e^2$.

Remarca.

Fie functia $f:\mathbf{R}\to\mathbf{R}$, cu $f(0)=-\lambda$, $\lambda>0$, care admite primitiva F.

Daca $f(x)+\lambda F(x)=\lambda+1, \forall x\in\mathbf{R}$ sa se calculeze $f(-1)$.

Marin Chirciu

Soluție.

Deducem ca $f(-1)=-\lambda e^{\lambda}$.

Problema144.

SP.542. If $z_1,z_2,z_3\in\mathbf{C}, |z_1|=|z_2|=|z_3|=1, z_1+z_2+z_3=1, z_1z_2z_3=1$ then find:

$$\Omega=\sum_{k=1}^{3}z_i^3\sum_{k=1}^{3}z_i^5\sum_{k=1}^{3}z_i^7.$$

RMM- Number 37, Summer 2025, Daniel Sitaru, Romania

Solutie.

Folosim $|z|^2=z\cdot\bar{z}, z\in\mathbf{C}$.

$$\text{Avem } 1=\bar{1}=\overline{z_1+z_2+z_3}=\bar{z_1}+\bar{z_2}+\bar{z_3}=\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}=\frac{z_1z_2+z_2z_3+z_3z_1}{z_1z_2z_3}=\frac{z_1z_2+z_2z_3+z_3z_1}{1}\Rightarrow$$

$\Rightarrow z_1z_2+z_2z_3+z_3z_1=1$.

Formam ecuatia de gradul al treilea cunoscand sumele lui Viete

$z_1+z_2+z_3=1, z_1z_2+z_2z_3+z_3z_1=1, z_1z_2z_3=1$:

Obtinem ecuatia $z^3-z^2+z-1=0 \Leftrightarrow (z-1)(z^2+1)=0$, cu solutiile $\{z_1,z_2,z_3\}=\{1,i,-i\}$.

Rezulta $\sum_{k=1}^{3} z_i^3=1, \sum_{k=1}^{3} z_i^5=1, \sum_{k=1}^{3} z_i^7=1$.

Deducem ca $\Omega=\sum_{k=1}^{3} z_i^3 \sum_{k=1}^{3} z_i^5 \sum_{k=1}^{3} z_i^7=1$.

Remarca.

If $z_1,z_2,z_3\in\mathbf{C}, |z_1|=|z_2|=|z_3|=1, z_1+z_2+z_3=1, z_1z_2z_3=1$ and $k,n,p\in\mathbf{N}$ then find:

$$\Omega=\sum_{k=1}^{3} z_i^{2k+1} \sum_{k=1}^{3} z_i^{2n+1} \sum_{k=1}^{3} z_i^{2p+1}.$$

Marin Chirciu

Solutie.

Folosim $|z|^2=z\cdot\bar{z}, z\in\mathbf{C}$.

Avem $1=\bar{1}=\overline{z_1+z_2+z_3}=\bar{z_1}+\bar{z_2}+\bar{z_3}=\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}=\frac{z_1z_2+z_2z_3+z_3z_1}{z_1z_2z_3}=\frac{z_1z_2+z_2z_3+z_3z_1}{1}\Rightarrow$

$\Rightarrow z_1z_2+z_2z_3+z_3z_1=1$.

Formam ecuatia de gradul al treilea cunoscand sumele lui Viete

$z_1+z_2+z_3=1, z_1z_2+z_2z_3+z_3z_1=1, z_1z_2z_3=1$:

Obtinem ecuatia $z^3-z^2+z-1=0 \Leftrightarrow (z-1)(z^2+1)=0$, cu solutiile $\{z_1,z_2,z_3\}=\{1,i,-i\}$.

Rezulta $\sum_{k=1}^{3} z_i^{2k+1}=1, \sum_{k=1}^{3} z_i^{2n+1}=1, \sum_{k=1}^{3} z_i^{2p+1}=1$.

Deducem ca $\Omega=\sum_{k=1}^{3} z_i^{2k+1} \sum_{k=1}^{3} z_i^{2n+1} \sum_{k=1}^{3} z_i^{2p+1}=1$.

Problema145.

JP.550. The non-coplanar points are given: A,B,C,D. If K -the midle of the segment $[BD]$, $(KM$ -bisector $\angle AKB, M\in(AB), (KP$ -bisector $\angle AKD, P\in(AD)$, and $N\in(AC)$, such that $\frac{AC}{AN}-\frac{BD}{2AK}=1$. Prove that

$$AN\cdot NC+AM\cdot MB\geq 2PD\cdot(AN+AM-AP).$$

RMM- Number 37, Summer 2025, Gheorghe Molea, Romania

Solutie.

Deoarece K este mijlocul lui $[BD]\Rightarrow\frac{BD}{2}=KB=KD\Rightarrow\frac{AC}{AN}-\frac{KB}{AK}=1$ si $\frac{AC}{AN}-\frac{KD}{AK}=1$.

Folosind teorema bisectoarei in ΔAKB si $\Delta AKD\Rightarrow\frac{AM}{MB}=\frac{AK}{KB}$ si $\frac{AP}{PD}=\frac{AK}{KD}$.

Din $\frac{AC}{AN}-\frac{KB}{AK}=1\Leftrightarrow\frac{AN+NC}{AN}-\frac{KB}{AK}=1\Leftrightarrow 1+\frac{NC}{AN}-\frac{KB}{AK}=1\Leftrightarrow\frac{NC}{AN}=\frac{KB}{AK}\Rightarrow$

$\Rightarrow\frac{NC}{AN}=\frac{KB}{AK}\overset{T-bis}{=}\frac{MB}{AM}\Rightarrow\frac{NC}{AN}=\frac{MB}{AM}\overset{R-Thales}{\Rightarrow}MN\parallel BC$, (1).

Analog:

Din $\frac{AC}{AN}-\frac{KD}{AK}=1\Leftrightarrow\frac{AN+NC}{AN}-\frac{KD}{AK}=1\Leftrightarrow 1+\frac{NC}{AN}-\frac{KD}{AK}=1\Leftrightarrow\frac{NC}{AN}=\frac{KD}{AK}\Rightarrow$

$\Rightarrow \frac{NC}{AN}=\frac{KD}{AK}\overset{T-bis}{=}\frac{PD}{AP}\Rightarrow \frac{NC}{AN}=\frac{PD}{AP}\overset{R-Thales}{\Rightarrow} PN\parallel DC$, (2).

Din $\frac{NC}{AN}=\frac{MB}{AM}$ si $\frac{NC}{AN}=\frac{PD}{AP}\Rightarrow \frac{MB}{AM}=\frac{PD}{AP}\overset{R-Thales}{\Rightarrow} PM\parallel BD$, (3).

Din (1), (2) si (3) $\Rightarrow \Delta MNP\sim\Delta BCD$.

Impartind ambii membri prin $AM\cdot MB$ obtinem:

$$\frac{AN\cdot NC+AM\cdot MB}{AM\cdot MB}\geq \frac{2PD\cdot(AN+AM-AP)}{AM\cdot MB}\Leftrightarrow \frac{AN}{AM}\cdot\frac{NC}{MB}+1\geq\frac{2PD}{MB}\left(\frac{AN}{AM}+1-\frac{AP}{AM}\right).$$

Folosind $\Delta MNP\sim\Delta BCD\Rightarrow \frac{AN}{AM}=\frac{AC}{AB},\frac{NC}{MB}=\frac{AC}{AB},\frac{PD}{MB}=\frac{AD}{AB}$ si $\frac{AP}{AM}=\frac{AD}{AB}$.

Egalitatile de mai sus inlocuite in inegalitatea ceruta conduc la:

$$\frac{AC}{AB}\cdot\frac{AC}{AB}+1\geq\frac{2AD}{AB}\left(\frac{AC}{AB}+1-\frac{AD}{AB}\right)\Leftrightarrow AB^2+AC^2\geq 2AD(AB+AC-AD)\Leftrightarrow$$

$$\Leftrightarrow AB^2+AC^2+2AD^2\geq 2AD(AB+AC)\Leftrightarrow$$

$$\Leftrightarrow \left(AB^2+AD^2\right)+\left(AC^2+AD^2\right)\geq 2AD(AB+AC)\Leftrightarrow (AB-AD)^2+(AC-AD)^2\geq 0,$$

cu egalitate pentru $AB=AC=AD$.

Remarca.

Fie tetraedrul $ABCD$, M -mijlocul muchiei BD, P,Q -picioarele bisectoarele unghiurilor AMB, respectiv AMD, iar $R\in(AC)$ astfel incat $\frac{AC}{AR}-\frac{MB}{AM}=1$. Aratati ca:

a) $(PQR)\parallel(BCD)$.

b) $\Delta PQR\sim\Delta BCD$.

Marin Chirciu

Solutie.

a) Folosind teorema bisectoarei in ΔAMB si $\Delta AMD\Rightarrow\frac{AM}{MB}=\frac{AP}{PB}$ si $\frac{AM}{MD}=\frac{AQ}{QD}$.

Deoarece M este mijlocul

lui $[BD]\Rightarrow MB=MD\Rightarrow\frac{AC}{AR}-\frac{MB}{AM}=1\Leftrightarrow\frac{AR+RC}{AR}-\frac{MB}{AM}=1\Leftrightarrow$

$\Leftrightarrow 1+\frac{RC}{AR}-\frac{MB}{AM}=1\Leftrightarrow\frac{RC}{AR}=\frac{MB}{AM}\Rightarrow\frac{RC}{AR}=\frac{MB}{AM}\overset{T-bis}{=}\frac{PB}{AP}\overset{R-Thales}{\Rightarrow}PR\parallel BC$, (1).

Analog:

$\frac{AC}{AR}-\frac{MD}{AM}=1\Leftrightarrow\frac{AR+RC}{AR}-\frac{MD}{AM}=1\Leftrightarrow$

$\Leftrightarrow 1+\frac{RC}{AR}-\frac{MD}{AM}=1\Leftrightarrow\frac{RC}{AR}=\frac{MD}{AM}\Rightarrow\frac{RC}{AR}=\frac{MD}{AM}\overset{T-bis}{=}\frac{DQ}{AQ}\overset{R-Thales}{\Rightarrow}QR\parallel DC$, (2).

Din (1) si (2) $\Rightarrow (PQR)\parallel(BCD)$

b) Din $\frac{RC}{AR}=\frac{MB}{AM}$ si $\frac{RC}{AR}=\frac{QD}{AQ}\Rightarrow\frac{PB}{AP}=\frac{QD}{AQ}\overset{R-Thales}{\Rightarrow}PQ\parallel BD$, (3).

Din (1), (2) si (3) $\Rightarrow\Delta PQR\sim\Delta BCD$.

Problema146.

UP.547. Find without any software:

$$\Omega=\int_1^2\frac{4x^4-6x-9}{x^4}e^{2x+\frac{3}{x}}dx.$$

RMM- Number 37, Summer 2025, Daniel Sitaru, Romania

Solutie.

Avem $\frac{4x^4-6x-9}{x^4}=4-\frac{6}{x^3}-\frac{9}{x^4}=\left(2-\frac{3}{x^2}\right)\left(2+\frac{3}{x^2}\right)-\frac{6}{x^3}$.

Obtinem:

$$\Omega=\int_1^2\frac{4x^4-6x-9}{x^4}e^{2x+\frac{3}{x}}dx=\int_1^2\left(\left(2-\frac{3}{x^2}\right)\left(2+\frac{3}{x^2}\right)-\frac{6}{x^3}\right)e^{2x+\frac{3}{x}}dx=$$

$$=\int_1^2\left(2-\frac{3}{x^2}\right)\left(2+\frac{3}{x^2}\right)e^{2x+\frac{3}{x}}dx-\int_1^2\frac{6}{x^3}e^{2x+\frac{3}{x}}dx=\int_1^2\left(2+\frac{3}{x^2}\right)\left(e^{2x+\frac{3}{x}}\right)'dx-\int_1^2\frac{6}{x^3}e^{2x+\frac{3}{x}}dx\overset{Parti}{=}$$

$$\overset{Parti}{=}\int_1^2\left(2+\frac{3}{x^2}\right)\left(e^{2x+\frac{3}{x}}\right)'dx-\int_1^2\frac{6}{x^3}e^{2x+\frac{3}{x}}dx=\left(2+\frac{3}{x^2}\right)\left(e^{2x+\frac{3}{x}}\right)\Big|_1^2-\int_1^2\left(2+\frac{3}{x^2}\right)'e^{2x+\frac{3}{x}}dx-\int_1^2\frac{6}{x^3}e^{2x+\frac{3}{x}}dx=$$

$$=\left(2+\frac{3}{x^2}\right)\left(e^{2x+\frac{3}{x}}\right)\Big|_1^2+\int_1^2\frac{6}{x^3}e^{2x+\frac{3}{x}}dx-\int_1^2\frac{6}{x^3}e^{2x+\frac{3}{x}}dx=\left(2+\frac{3}{x^2}\right)\left(e^{2x+\frac{3}{x}}\right)\Big|_1^2=$$

$$=\left(2+\frac{3}{4}\right)\left(e^{4+\frac{3}{2}}\right)-\left(2+\frac{3}{1}\right)\left(e^{2+\frac{3}{1}}\right)=\frac{11}{4}e^{\frac{11}{2}}-5e^5.$$

Remarca.

Let $a,b\in\mathbf{R}^*$. Find :

$$\Omega=\int_1^2\frac{a^2x^4-2bx-b^2}{x^4}e^{ax+\frac{b}{x}}dx.$$

Marin Chirciu

Solutie.

Avem $\frac{a^2x^4-2bx-b^2}{x^4}=a^2-\frac{2b}{x^3}-\frac{b^2}{x^4}=\left(a-\frac{b}{x^2}\right)\left(a+\frac{b}{x^2}\right)-\frac{2b}{x^3}$.

Obtinem:

$$\Omega=\int_1^2\frac{a^2x^4-2bx-b^2}{x^4}e^{ax+\frac{b}{x}}dx=\int_1^2\left(\left(a-\frac{b}{x^2}\right)\left(a+\frac{b}{x^2}\right)-\frac{2b}{x^3}\right)e^{ax+\frac{b}{x}}dx=$$

$$=\left(a+\frac{b}{4}\right)\left(e^{2a+\frac{b}{2}}\right)-\left(a+\frac{b}{1}\right)\left(e^{a+\frac{b}{1}}\right)=\frac{4a+b}{4}e^{\frac{4a+b}{2}}-(a+b)e^{a+b}.$$

Problema147.

UP.546. Prove without any software:

$$\int_0^2\sqrt[3]{\frac{x}{2}+\sqrt{\frac{1}{27}+\frac{x^2}{4}}}dx+\int_0^2\sqrt[3]{\frac{x}{2}-\sqrt{\frac{1}{27}+\frac{x^2}{4}}}dx>\frac{5}{4}.$$

RMM- Number 37, Summer 2025, Daniel Sitaru, Romania

Solutie.

Folosim **formulele lui Cardano** pentru ecuația de gradul al treilea $x^3+px+q=0$.

Discriminantul este $\Delta=-\left(4p^3+27q^2\right)$.

Dacă $\Delta=-\left(4p^3+27q^2\right)<0$, ecuația are o singură rădăcină reală

$x_1=\sqrt[3]{\frac{-q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{\frac{-q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$ și $x_{2,3}\in\mathbf{C}-\mathbf{R}$.

Pentru $p=1,q=-1$ ecuația de gradul al treilea $x^3+x-1=0$ are $\Delta=-(4+27)<0\Rightarrow$

$x_1=\sqrt[3]{\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{27}}}+\sqrt[3]{\frac{1}{2}-\sqrt{\frac{1}{4}+\frac{1}{27}}}$ şi $x_{2,3}\in\mathbf{C}-\mathbf{R}$.

Avem:

$$f(x)=\sqrt[3]{\frac{x}{2}+\sqrt{\frac{1}{27}+\frac{x^2}{4}}}+\sqrt[3]{\frac{x}{2}-\sqrt{\frac{1}{27}+\frac{x^2}{4}}}\ge f(1)=\sqrt[3]{\frac{1}{2}+\sqrt{\frac{1}{27}+\frac{1}{4}}}+\sqrt[3]{\frac{1}{2}-\sqrt{\frac{1}{27}+\frac{1}{4}}}=x_1.$$

Aratam ca $x_1>\frac{5}{8}$.

Consideram functia $g:[0,6]\to\mathbf{R}$, $g(x)=x^3+x-1$; avem $g'(x)=3x^2+1>0\Rightarrow g\uparrow$.

Deoarece $g\left(\frac{5}{8}\right)=\left(\frac{5}{8}\right)^3+\frac{5}{8}-1=\frac{-323}{512}<0$ si $g(1)=1>0\Rightarrow\frac{5}{8}<x_1<1$.

Obtinem:

$$\int_0^2\sqrt[3]{\frac{x}{2}+\sqrt{\frac{1}{27}+\frac{x^2}{4}}}dx+\int_0^2\sqrt[3]{\frac{x}{2}-\sqrt{\frac{1}{27}+\frac{x^2}{4}}}dx=\int_0^2\left(\sqrt[3]{\frac{x}{2}+\sqrt{\frac{1}{27}+\frac{x^2}{4}}}+\sqrt[3]{\frac{x}{2}-\sqrt{\frac{1}{27}+\frac{x^2}{4}}}\right)dx>$$

$$>\int_0^2 x_1dx>\int_0^2\frac{5}{8}dx=\frac{5}{8}x\Big|_0^2=\frac{5}{8}\cdot 2=\frac{5}{4}.$$

Problema148.

Evaluate

$$\int_2^4\frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(3+x)}}dx.$$

About Mathematics, 4/2024

Solutie.

$$I=\int_2^4\frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(3+x)}}dx\overset{x=6-t}{=}\int_4^2\frac{\sqrt{\ln(3+t)}}{\sqrt{\ln(3+t)}+\sqrt{\ln(9-t)}}(-dt)=\int_2^4\frac{\sqrt{\ln(3+x)}}{\sqrt{\ln(3+x)}+\sqrt{\ln(9-x)}}dx$$

$$2I=I+I=\int_2^4\frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(3+x)}}dx+\int_2^4\frac{\sqrt{\ln(3+x)}}{\sqrt{\ln(3+x)}+\sqrt{\ln(9-x)}}dx=\int_2^4 1\cdot dx=x\Big|_2^4=4-2=2.$$

Obtinem $2I=2\Leftrightarrow I=1$.

Deducem ca $\int_2^4\frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(3+x)}}dx=1$.

Remarca.

Let $a\ge 1$. Evaluate

$$\int_{a-1}^{a+1}\frac{\sqrt{\ln(3a-x)}}{\sqrt{\ln(3a-x)}+\sqrt{\ln(a+x)}}dx.$$

Marin Chirciu

Solutie.

Deducem ca $\int_{a-1}^{a+1}\frac{\sqrt{\ln(3a-x)}}{\sqrt{\ln(3a-x)}+\sqrt{\ln(a+x)}}dx=1$.

Remarca.

Let $f:[a-1,a+1]\to(0,\infty)$, f -continua. Evaluate

$$\int_{a-1}^{a+1} \frac{f(3a-x)}{f(3a-x)+f(a+x)}dx.$$

Marin Chirciu

Solutie.

Deducem ca $\int_{a-1}^{a+1} \frac{f(3a-x)}{f(3a-x)+f(a+x)}dx=1$.

Problema149.

4928. If AD, BE, CF -internal bisectors, r_A, r_B, r_C -inradii of triangles AEF, BFD, CDE then:

$$r_A^2+r_B^2+r_C^2 \le \frac{3R^4}{64r^2}$$

Crux Mathematicorum, March 2024, George Apostolopoulos, Greece

Solutie.

Lema.

If AD, BE, CF -internal bisectors, r_A, r_B, r_C -inradii of triangles AEF, BFD, CDE then:

$$r_A^2 \le \frac{F\cdot bc}{3\sqrt{3}(a+b)(a+c)}.$$

Solutie.

Folosim $r_A=\frac{S_{AEF}}{p_{AEF}}=\frac{F'}{p'}$, unde:

$$S_{AEF}=\frac{AE\cdot AF\cdot \sin A}{2}=\frac{\frac{bc}{a+c}\cdot\frac{bc}{a+b}\cdot\frac{a}{2R}}{2}=\frac{abc}{4R}\cdot\frac{bc}{(a+b)(a+c)}=\frac{4RS}{4R}\cdot\frac{bc}{(a+b)(a+c)}=$$

$$=\frac{S\cdot bc}{(a+b)(a+c)}.$$

$$p_{AEF}=\frac{AE+AF+EF}{2}.$$

$$r_A^2=\frac{(F')^2}{(p')^2}\overset{Hadwiger}{\le}\frac{(F')^2}{(p')^2}=\frac{(F')^2}{3F'\sqrt{3}}=\frac{F'}{3\sqrt{3}}=\frac{\frac{F\cdot bc}{(a+b)(a+c)}}{3\sqrt{3}}=\frac{F\cdot bc}{3\sqrt{3}(a+b)(a+c)}.$$

Am folosit inegalitatea lui Hadwiger $p^2\ge 3F\sqrt{3}$.

$$LHS=r_A^2+r_B^2+r_C^2\overset{Lema}{\le}\sum\frac{F\cdot bc}{3\sqrt{3}(a+b)(a+c)}=\frac{F}{3\sqrt{3}}\sum\frac{bc}{(a+b)(a+c)}=$$

$$=\frac{F}{3\sqrt{3}}\cdot\frac{p^2+r^2-2Rr}{p^2+r^2+2Rr}\overset{(1)}{\le}\frac{F}{3\sqrt{3}}\cdot\frac{3R}{8r}\overset{(2)}{\le}\frac{3R^4}{64r^2}=RHS,$$

unde $(1)\Leftrightarrow \frac{p^2+r^2-2Rr}{p^2+r^2+2Rr}\le\frac{3R}{8r} \Leftrightarrow 8r\left(p^2+r^2-2Rr\right)\le 3R\left(p^2+r^2+2Rr\right)\Leftrightarrow$

$\Leftrightarrow p^2(3R-8r)+r\left(6R^2+19Rr-8r^2\right)\ge 0$.

Distingem cazurile:

Cazul1).Daca $(3R-8r)\ge 0$, inegalitatea este evidenta.

Cazul2).Daca $(3R-8r)<0$, inegalitatea se rescrie: $r\left(6R^2+19Rr-8r^2\right)\ge p^2(8r-3R)$, care rezulta din inegalitatea lui Gerretsen: $p^2\le 4R^2+4Rr+3r^2$.

Ramane sa aratam ca:

$r\left(6R^2+19Rr-8r^2\right)\geq\left(4R^2+4Rr+3r^2\right)\left(8r-3R\right)\Leftrightarrow 6R^3-7R^2r-2Rr^2-16r^3\geq 0\Leftrightarrow$

$\Leftrightarrow\left(R-2r\right)\left(6R^2+5Rr+8r^2\right)\geq 0$, vezi $R\geq 2r$,(Euler).

Inegalitatea(2) $\Leftrightarrow\frac{F}{3\sqrt{3}}\cdot\frac{3R}{8r}\leq\frac{3R^4}{64r^2}\Leftrightarrow\frac{pr}{3\sqrt{3}}\cdot\frac{3R}{8r}\leq\frac{3R^4}{64r^2}\Leftrightarrow\frac{p}{3\sqrt{3}}\leq\frac{R^3}{8r^2}$, care rezulta din inegalitatea lui Mitrinovic $p\leq\frac{3R\sqrt{3}}{2}$.

Ramane sa aratam ca $\frac{\frac{3R\sqrt{3}}{2}}{3\sqrt{3}}\leq\frac{R^3}{8r^2}\Leftrightarrow 4r^2\leq R^2$,vezi(Euler).

Egalitatea are loc dacă şi numai dacă triunghiul este echilateral.

Remarca.

If AD, BE, CF -internal bisectors, r_A, r_B, r_C -inradii of triangles AEF, BFD, CDE then:

$$1).\ r_Ar_Br_C\leq\left(\frac{F}{12\sqrt{3}}\right)^{\frac{3}{2}}.$$

$$2).\ r_A^2\leq\frac{F}{24\sqrt{3}}\cdot\frac{b+c}{a}.$$

Dezvoltări, Marin Chirciu

Problema150.

UP.542. Find :

$$\Omega=\sum_{n=1}^{\infty}\arctan\frac{49}{49+(7n+1)(7n+8)}.$$

RMM- Number 37, Summer 2025, Daniel Sitaru, Romania

Solutie.

Folosind $\arctan\frac{1}{x}-\arctan\frac{1}{x+1}=\arctan\frac{1}{x^2+x+1}$, $x>0$, pentru $x=n+\frac{1}{7}$ obtinem:

$$\arctan\frac{1}{n+\frac{1}{7}}-\arctan\frac{1}{n+\frac{1}{7}+1}=\arctan\frac{1}{\left(n+\frac{1}{7}\right)^2+n+\frac{1}{7}+1}=\frac{49}{49+(7n+1)(7n+8)}.$$

Rezulta:

$$\Omega=\sum_{n=1}^{\infty}\arctan\frac{49}{49+(7n+1)(7n+8)}=\sum_{n=1}^{\infty}\left(\arctan\frac{1}{n+\frac{1}{7}}-\arctan\frac{1}{n+\frac{1}{7}+1}\right)=$$

$$=\arctan\frac{1}{1+\frac{1}{7}}-\arctan\frac{1}{2+\frac{1}{7}}+\arctan\frac{1}{2+\frac{1}{7}}-\arctan\frac{1}{3+\frac{1}{7}}+\ldots+\arctan\frac{1}{n+\frac{1}{7}}-\arctan\frac{1}{n+1+\frac{1}{7}}+\ldots=$$

$$=\lim_{n\to\infty}\left(\arctan\frac{1}{1+\frac{1}{7}}-\arctan\frac{1}{n+1+\frac{1}{7}}\right)=\arctan\frac{7}{8}.$$

Remarca.

Let $\lambda>0$. Find :

$$\Omega=\sum_{n=1}^{\infty}\arctan\frac{\lambda^2}{\lambda^2+(\lambda n+1)(\lambda n+\lambda+1)}.$$

Marin Chirciu

Solutie.

Folosind $\arctan\frac{1}{x}-\arctan\frac{1}{x+1}=\arctan\frac{1}{x^2+x+1}$, $x>0$, pentru $x=n+\frac{1}{\lambda}$ obtinem:

$$\arctan\frac{1}{n+\frac{1}{\lambda}}-\arctan\frac{1}{n+\frac{1}{\lambda}+1}=\arctan\frac{1}{\left(n+\frac{1}{\lambda}\right)^2+n+\frac{1}{\lambda}+1}=\frac{\lambda^2}{\lambda^2+(\lambda n+1)(\lambda n+\lambda+1)}.$$

Rezulta:

$$\Omega=\sum_{n=1}^{\infty}\arctan\frac{\lambda^2}{\lambda^2+(\lambda n+1)(\lambda n+\lambda+1)}=\sum_{n=1}^{\infty}\left(\arctan\frac{1}{n+\frac{1}{\lambda}}-\arctan\frac{1}{n+\frac{1}{\lambda}+1}\right)=$$

$$=\arctan\frac{1}{1+\frac{1}{\lambda}}-\arctan\frac{1}{2+\frac{1}{\lambda}}+\arctan\frac{1}{2+\frac{1}{\lambda}}-\arctan\frac{1}{3+\frac{1}{\lambda}}+\ldots+\arctan\frac{1}{n+\frac{1}{\lambda}}-\arctan\frac{1}{n+1+\frac{1}{\lambda}}+\ldots=$$

$$=\lim_{n\to\infty}\left(\arctan\frac{1}{1+\frac{1}{\lambda}}-\arctan\frac{1}{n+1+\frac{1}{\lambda}}\right)=\arctan\frac{\lambda}{\lambda+1}.$$

Problema151.

SP.541. Find all continous functions $f:\mathbf{R}\to\mathbf{R}$ such that:

$$f\left(\frac{x}{3}\right)-3f(x)=15x,(\forall)x\in\mathbf{R}.$$

RMM- Number 37, Summer 2025, Daniel Sitaru, Romania

Solutie.

$$f\left(\frac{x}{3}\right)-3f(x)=15x\Leftrightarrow f(x)=\frac{1}{3}f\left(\frac{x}{3}\right)-5x.$$

Dand lui x valorile: $\frac{x}{3},\frac{x}{3^2},\ldots,\frac{x}{3^n}$ obtinem succesiv:

$$f(x)=\frac{1}{3}f\left(\frac{x}{3}\right)-5x=\frac{1}{3}\left[\frac{1}{3}f\left(\frac{x}{3^2}\right)-5\frac{x}{3}\right]-5x=\frac{1}{3^2}f\left(\frac{x}{3^2}\right)-5x\left(1+\frac{1}{3^2}\right)=$$

$$=\frac{1}{3^2}\left[\frac{1}{3}f\left(\frac{x}{3^3}\right)-5\frac{x}{3^2}\right]-5x\left(1+\frac{1}{3^2}\right)=\frac{1}{3^3}f\left(\frac{x}{3^3}\right)-5\frac{x}{3^4}-5x\left(1+\frac{1}{3^2}\right)=$$

$$=\frac{1}{3^3}f\left(\frac{x}{3^3}\right)-5x\left(1+\frac{1}{3^2}+\frac{1}{3^4}\right)=\ldots=\frac{1}{3^n}f\left(\frac{x}{3^n}\right)-5x\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\ldots+\frac{1}{3^{2n}}\right).$$

Trecand la limita pentru $n\to\infty$ si tinand seama ca functia este continua si $f(0)=0$ obtinem

$$f(x)=-5x\lim_{n\to\infty}\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\ldots+\frac{1}{3^{2n}}\right)=-5x\cdot\frac{9}{8}=\frac{-45}{8}x.$$

Deducem ca functia cautata este $f:\mathbf{R}\to\mathbf{R}$, $f(x)=\frac{-45}{8}x$.

Remarca.

Let $a>1, b\in\mathbf{R}$ fixed. Find all continous functions $f:\mathbf{R}\to\mathbf{R}$ such that:

$$f\left(\frac{x}{a}\right)-af\left(x\right)=abx,\left(\forall\right)x\in\mathbf{R}.$$

Marin Chirciu

Solutie.

$f\left(\frac{x}{a}\right)-af\left(x\right)=abx \Leftrightarrow f\left(x\right)=\frac{1}{a}f\left(\frac{x}{a}\right)-bx.$

Dand lui x valorile: $\frac{x}{a},\frac{x}{a^2},...,\frac{x}{a^n}$ obtinem succesiv:

$$f\left(x\right)=\frac{1}{a}f\left(\frac{x}{a}\right)-bx=\frac{1}{a}\left[\frac{1}{a}f\left(\frac{x}{a^2}\right)-b\frac{x}{a}\right]-bx=\frac{1}{a^2}f\left(\frac{x}{a^2}\right)-ax\left(1+\frac{1}{a^2}\right)=$$

$$=\frac{1}{a^2}\left[\frac{1}{a}f\left(\frac{x}{a^3}\right)-b\frac{x}{a^2}\right]-bx\left(1+\frac{1}{a^2}\right)=\frac{1}{a^3}f\left(\frac{x}{a^3}\right)-b\frac{x}{a^4}-bx\left(1+\frac{1}{a^2}\right)=$$

$$=\frac{1}{a^3}f\left(\frac{x}{a^3}\right)-bx\left(1+\frac{1}{a^2}+\frac{1}{a^4}\right)=...=\frac{1}{a^n}f\left(\frac{x}{a^n}\right)-bx\left(1+\frac{1}{a^2}+\frac{1}{a^4}+...+\frac{1}{a^{2n}}\right).$$

Trecand la limita pentru $n\to\infty$ si tinand seama ca functia este continua si $f\left(0\right)=0$ obtinem

$$f\left(x\right)=-bx\lim_{n\to\infty}\left(1+\frac{1}{a^2}+\frac{1}{a^4}+...+\frac{1}{a^{2n}}\right)=-bx\cdot\frac{a^2}{a^2-1}=\frac{a^2b}{1-a^2}x.$$

Deducem ca functia cautata este $f:\mathbf{R}\to\mathbf{R}$, $f\left(x\right)=\frac{a^2b}{1-a^2}x$.

Problema152.

UP.541. Find:

$$\Omega=\int_1^{\sqrt{3}}\frac{x-\arctan x}{\left(1+x^2\right)^2\left(\arctan x\right)^3}dx.$$

RMM- Number 37, Summer 2025, Daniel Sitaru, Romania

Solutie.

Folosind substitutia $t=\arctan x \Rightarrow x=\tan t \Rightarrow dx=\left(1+\tan^2 t\right)dt \Rightarrow$

$$\Omega=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\tan t-t}{\left(1+\tan^2 t\right)^2 t^3}\left(1+\tan^2 t\right)dt=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\tan t-t}{\left(1+\tan^2 t\right)t^3}dt=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\tan t}{\left(1+\tan^2 t\right)t^3}dt-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{t}{\left(1+\tan^2 t\right)t^3}dt=$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\frac{\sin t}{\cos t}}{\frac{1}{\cos^2 t}t^3}dt-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{t}{\frac{1}{\cos^2 t}t^3}dt=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin t\cos t}{t^3}dt-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos^2 t}{t^2}dt=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sin 2t}{t^3}dt-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1+\cos 2t}{2t^2}dt=$$

$$=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{-1}{2t^2}\right)'\sin 2t dt-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{2t^2}dt-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos 2t}{2t^2}dt=\frac{1}{2}\left[\left(\frac{-1}{2t^2}\right)\sin 2t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{-1}{2t^2}\right)\cdot 2\cos 2t dt\right]-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{2t^2}dt-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos 2t}{2t^2}dt=$$

$$=\frac{1}{2}\left[\left(\frac{-1}{2\left(\frac{\pi}{3}\right)^2}\right)\sin\frac{2\pi}{3}+\left(\frac{1}{2\left(\frac{\pi}{4}\right)^2}\right)\sin\frac{2\pi}{4}+\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{t^2}\cos 2t dt\right]+\frac{1}{2t}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}}-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos 2t}{2t^2}dt=$$

$$=\frac{1}{2}\left[\left(\frac{-1}{2\left(\frac{\pi}{3}\right)^2}\right)\sin\frac{2\pi}{3}+\left(\frac{1}{2\left(\frac{\pi}{4}\right)^2}\right)\sin\frac{2\pi}{4}+\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{t^2}\cos 2t dt\right]+\frac{1}{2\cdot\frac{\pi}{3}}-\frac{1}{2\cdot\frac{\pi}{4}}-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos 2t}{2t^2}dt=$$

$$=\frac{1}{2}\left(\frac{-1}{2\left(\frac{\pi}{3}\right)^2}\right)\cdot\frac{\sqrt{3}}{2}+\frac{1}{2}\left(\frac{1}{2\left(\frac{\pi}{4}\right)^2}\right)\cdot 1+\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{t^2}\cos 2t dt+\frac{1}{\frac{2\pi}{3}}-\frac{1}{\frac{2\pi}{4}}-\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos 2t}{t^2}dt=$$

$$=\frac{-\sqrt{3}}{8\left(\frac{\pi}{3}\right)^2}+\frac{1}{4\left(\frac{\pi}{4}\right)^2}+\frac{3}{2\pi}-\frac{4}{2\pi}=\frac{-\sqrt{3}}{8\frac{\pi^2}{9}}+\frac{1}{4\frac{\pi^2}{16}}-\frac{1}{2\pi}=\frac{-9\sqrt{3}}{8\pi^2}+\frac{4}{\pi^2}-\frac{1}{2\pi}.$$

Deducem ca $\Omega=\int_1^{\sqrt{3}}\frac{x-\arctan x}{\left(1+x^2\right)^2\left(\arctan x\right)^3}dx=\frac{4}{\pi^2}-\frac{9\sqrt{3}}{8\pi^2}-\frac{1}{2\pi}$.

Remarca.
If $k\in\mathbf{N}$ find:

$$\Omega=\int_1^{\sqrt{3}}\frac{kx-\arctan x}{\left(1+x^2\right)^2\left(\arctan x\right)^{2k+1}}dx.$$

Marin Chirciu

Solutie.

Pentru $k=0$ avem $\Omega=\int_1^{\sqrt{3}}\frac{-\arctan x}{\left(1+x^2\right)^2\left(\arctan x\right)}dx=-\int_1^{\sqrt{3}}\frac{1}{\left(1+x^2\right)^2}dx$.

Integrand prin parti avem

$$\int\frac{1}{x^2+1}dx=\int x'\frac{1}{x^2+1}dx=x\cdot\frac{1}{x^2+1}-\int x\left(\frac{1}{x^2+1}\right)'dx=\frac{x}{x^2+1}-\int x\cdot\frac{-2x}{\left(x^2+1\right)^2}dx=$$

$$=\frac{x}{x^2+1}+2\int\frac{x^2}{\left(x^2+1\right)^2}dx=\frac{x}{x^2+1}+2\int\frac{x^2+1-1}{\left(x^2+1\right)^2}dx=\frac{x}{x^2+1}+2\int\frac{1}{x^2+1}-2\int\frac{1}{\left(x^2+1\right)^2}dx.$$

Obținem $\Omega=-\int_1^{\sqrt{3}}\frac{1}{\left(1+x^2\right)^2}dx=-\left(\frac{\sqrt{3}}{8}-\frac{1}{4}+\frac{\pi}{24}\right)=\frac{1}{4}-\frac{\sqrt{3}}{8}-\frac{\pi}{24}$.

In continuare fie $k\geq 1$.

Folosind substitutia $t=\arctan x\Rightarrow x=\tan t\Rightarrow dx=\left(1+\tan^2 t\right)dt\Rightarrow$

$$\Omega=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{k\tan t-t}{\left(1+\tan^2 t\right)^2 t^{2k+1}}\left(1+\tan^2 t\right)dt=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{k\tan t-t}{\left(1+\tan^2 t\right)t^{2k+1}}dt=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{k\tan t}{\left(1+\tan^2 t\right)t^{2k+1}}dt-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{t}{\left(1+\tan^2 t\right)t^{2k+1}}dt=$$

$$=\frac{-\sqrt{3}\cdot 3^{2k}}{8\pi^{2k}}+\frac{4^{2k-1}}{\pi^{2k}}+\frac{3^{2k-1}-4^{2k-1}}{2(2k-1)\pi^{2k-1}}.$$

Deducem ca $\Omega=\int_1^{\sqrt{3}}\frac{kx-\arctan x}{\left(1+x^2\right)^2(\arctan x)^{2k+1}}dx=\frac{4^{2k-1}}{\pi^{2k}}-\frac{\sqrt{3}\cdot 3^{2k}}{8\pi^{2k}}+\frac{3^{2k-1}-4^{2k-1}}{2(2k-1)\pi^{2k-1}}$.

Problema153.

JP.552. Justify if exists non-zeo natural numbers a,b,c,d , different in pairs, such that we have :

$$a(b+c-a)=b(a+c-b)=c(a+b-c)=\frac{a+b+c}{d}.$$

RMM- Number 37, Summer 2025, Gheorghe Molea, Romania

Solutie.

Avem $a(b+c-a)=b(a+c-b)=c(a+b-c)=\frac{b+c-a}{\frac{1}{a}}=\frac{a+c-b}{\frac{1}{b}}=\frac{a+b-c}{\frac{1}{c}}=$

$$\frac{(a+b-c)+(a+c-b)+(a+b-c)}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=\frac{a+b+c}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=\frac{a+b+c}{d}$$, daca $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=d$,

vezi $(a,b,c,d)=(2,3,6,1)$, deoarece $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$.

Remarca.

Justify if exists non-zeo natural numbers a,b,c,d,e, different in pairs, such that we have :

$$a(b+c+d-a)=b(a+c+d-b)=c(a+b+d-c)=d(a+b+c-d)=\frac{2(a+b+c+d)}{e}.$$

Marin Chirciu

Solutie.

Avem $a(b+c+d-a)=b(a+c+d-b)=c(a+b+d-c)=d(a+b+c-d)=$

$$=\frac{b+c+d-a}{\frac{1}{a}}=\frac{a+c+d-b}{\frac{1}{b}}=\frac{a+b+d-c}{\frac{1}{c}}=\frac{a+b+c-d}{\frac{1}{d}}=$$

$$=\frac{(b+c+d-a)+(a+c+d-b)+(a+b+d-c)+(a+b+c-d)}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}=$$

$$=\frac{2(a+b+c+d)}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}=\frac{2(a+b+c+d)}{e}$$, daca $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=e$,

vezi $(a,b,c,d,e)=(2,4,6,12,1)$, deoarece $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{12}=1$.

Problema154.

JP.551. Find the real numbers x,y,z knowing that they meet the conditions :

$$\begin{cases} x+y+z=1 \\ xy+(x+y)(z+1)=\frac{4}{3} \end{cases}.$$

RMM- Number 37, Summer 2025, Gheorghe Molea, Romania

Solutie.

Notand $x+y=S$ si $xy=P$ avem $\begin{cases} S=1-z \\ P+S(z+1)=\frac{4}{3} \end{cases} \Leftrightarrow \begin{cases} S=1-z \\ P=\frac{1}{3}+z^2 \end{cases}.$

Formam ecuatia de gradul al doilea cunoscand suma $S=1-z$ si produsul $P=\frac{1}{3}+z^2$.

Obtinem $t^2-St+P=0 \Leftrightarrow t^2-(1-z)t+\frac{1}{3}+z^2=0 \Leftrightarrow 3t^2-3(1-z)t+3z^2+1=0$,

cu $\Delta=9(z-1)^2-12(3z^2+1)=-27t^2-18z-3=-3(3z+1)^2$.

Deoarece solutiile x, y ale ecuatiei de gradul al doilea trebuie sa fie reale punem conditia

$\Delta=-3(3z+1)^2\geq 0 \Leftrightarrow z=\frac{-1}{3}$.

Punand $z=\frac{-1}{3}$ in ecuatia $3t^2-3(1-z)t+3z^2+1=0$ obtinem $9t^2-12t+4=0 \Leftrightarrow (3t-2)^2=0$

$\Leftrightarrow t=\frac{2}{3}$, de unde $x=y=\frac{2}{3}$.

Deducem ca $(x,y,z)=\left(\frac{2}{3},\frac{2}{3},\frac{-1}{3}\right)$ este solutia reala unica a sistemului.

Remarca.

Let $\lambda>0$ fixed. Solve in real numbers x, y, z the system:

$$\begin{cases} x+y+z=\lambda \\ xy+(x+y)(z+\lambda)=\frac{4\lambda^2}{3} \end{cases}.$$

Marin Chirciu

Solutie.

Notand $x+y=S$ si $xy=P$ avem $\begin{cases} S=\lambda-z \\ P+S(z+\lambda)=\frac{4\lambda^2}{3} \end{cases} \Leftrightarrow \begin{cases} S=\lambda-z \\ P=\frac{\lambda^2}{3}+z^2 \end{cases}.$

Formam ecuatia de gradul al doilea cunoscand suma $S=\lambda-z$ si produsul $P=\frac{\lambda^2}{3}+z^2$.

Obtinem $t^2-St+P=0 \Leftrightarrow t^2-(\lambda-z)t+\frac{\lambda^2}{3}+z^2=0 \Leftrightarrow 3t^2-3(\lambda-z)t+3z^2+\lambda^2=0$,

cu $\Delta=9(z-\lambda)^2-12(3z^2+\lambda^2)=-27t^2-18\lambda z-3\lambda^2=-3(3z+\lambda)^2$.

Deoarece solutiile x, y ale ecuatiei de gradul al doilea trebuie sa fie reale punem conditia

$\Delta=-3(3z+\lambda)^2\geq 0 \Leftrightarrow z=\frac{-\lambda}{3}$.

Punand $z=\frac{-\lambda}{3}$ in ecuatia $3t^2-3(\lambda-z)t+3z^2+\lambda^2=0$ obtinem $9t^2-12\lambda t+4\lambda^2=0 \Leftrightarrow$

$\Leftrightarrow (3t-2\lambda)^2=0 \Leftrightarrow t=\frac{2\lambda}{3}$, de unde $x=y=\frac{2\lambda}{3}$.

Deducem ca $(x,y,z)=\left(\frac{2\lambda}{3},\frac{2\lambda}{3},\frac{-\lambda}{3}\right)$ este solutia reala unica a sistemului.

Problema155.

UP.549. If $0<a\le b$ then find:

$$\Omega(a,b)=\int_a^b \frac{\ln x}{x^2+(a+b)x+ab}dx.$$

RMM- Number 37, Summer 2025, Daniel Sitaru, Romania

Solutie.

Folosind substitutia $x=\frac{ab}{t} \Rightarrow dx=\frac{-ab}{t^2}dt \Rightarrow$

$$\Rightarrow \Omega(a,b)=\int_a^b \frac{\ln x}{x^2+(a+b)x+ab}dx=\int_b^a \frac{\ln\frac{ab}{t}}{\left(\frac{ab}{t}\right)^2+(a+b)\frac{ab}{t}+ab}\left(-\frac{ab}{t^2}dt\right)=$$

$$=\int_a^b \frac{\ln(ab)-\ln t}{t^2+(a+b)t+ab}dt=\int_a^b \frac{\ln(ab)}{t^2+(a+b)t+ab}dt-\int_a^b \frac{\ln t}{t^2+(a+b)t+ab}dt=$$

$$=\int_a^b \frac{\ln(ab)}{t^2+(a+b)t+ab}dt-\Omega(a,b).$$

Rezulta ca $\Omega(a,b)=\int_a^b \frac{\ln(ab)}{t^2+(a+b)t+ab}dt-\Omega(a,b) \Leftrightarrow 2\Omega(a,b)=\int_a^b \frac{\ln(ab)}{t^2+(a+b)t+ab}dt \Leftrightarrow$

$$\Leftrightarrow \Omega(a,b)=\frac{\ln(ab)}{2}\int_a^b \frac{1}{t^2+(a+b)t+ab}dt.$$

Sa calculam $\int_a^b \frac{1}{t^2+(a+b)t+ab}dt$.

Avem

$$\int_a^b \frac{1}{t^2+(a+b)t+ab}dt=\int_a^b \frac{1}{(t+a)(t+b)}dt=\frac{1}{b-a}\int_a^b\left(\frac{1}{t+a}-\frac{1}{t+b}\right)dt=\frac{1}{b-a}\left(\ln(t+a)-\ln(t+b)\right)=$$

$$=\frac{1}{b-a}\ln\left(\frac{t+a}{t+b}\right)\Bigg|_a^b=\frac{1}{b-a}\left(\ln\left(\frac{b+a}{b+b}\right)-\ln\left(\frac{a+a}{a+b}\right)\right)=\frac{1}{b-a}\ln\frac{(a+b)^2}{4ab}, \text{ pentru } a<b.$$

Pentru $a=b$ avem $\int_a^b \frac{1}{t^2+(a+b)t+ab}dt=0$.

Deducem ca:

$$\Omega(a,b)=\frac{\ln(ab)}{2}\int_a^b \frac{1}{t^2+(a+b)t+ab}dt=\frac{\ln(ab)}{2}\cdot\frac{1}{b-a}\ln\frac{(a+b)^2}{4ab}=\frac{\ln(ab)}{2(b-a)}\ln\frac{(a+b)^2}{4ab},$$

pentru $a<b$ si $\Omega(a,b)=0$ pentru $a=b$.

Remarca.

If $0<a\le b$ then find:

$$\Omega(a,b)=\int_a^b \frac{\ln x}{(x+a)^2}dx.$$

Marin Chirciu

Solutie.

Folosind substitutia $x=\frac{a^2}{t} \Rightarrow dx=\frac{-a^2}{t^2}dt \Rightarrow$

$$\Omega(a,b)=\int_a^b \frac{\ln x}{(x+a)^2}dx=\int_b^a \frac{\ln\frac{a^2}{t}}{\left(\frac{a^2}{t}+a\right)^2}\left(-\frac{a^2}{t^2}dt\right)=\int_a^b \frac{\ln(a^2)-\ln t}{(t+a)^2}dt=\int_a^b \frac{\ln(a^2)}{(t+a)^2}dt-\int_a^b \frac{\ln t}{(t+a)^2}dt=$$

$$=\int_a^b \frac{2\ln a}{(t+a)^2}dt-\Omega(a,b).$$

Rezulta ca $\Omega(a,b)=\int_a^b \frac{2\ln a}{(t+a)^2}dt-\Omega(a,b) \Leftrightarrow 2\Omega(a,b)=2\ln a\int_a^b \frac{1}{(t+a)^2}dt \Leftrightarrow$

$$\Leftrightarrow \Omega(a,b)=\ln a\cdot\int_a^b \frac{1}{(t+a)^2}dt$$

Sa calculam $\int_a^b \frac{1}{(t+a)^2}dt$.

Avem $\int_a^b \frac{1}{(t+a)^2}dt=\left.\frac{-1}{t+a}\right|_a^b=\frac{-1}{b+a}+\frac{1}{2a}=\frac{-2a+b+a}{2a(a+b)}=\frac{b-a}{2a(a+b)}$.

Deducem ca: $\Omega(a,b)=\ln a\cdot\frac{b-a}{2a(a+b)}=\frac{(b-a)\ln a}{2a(a+b)}$.

Problema156.

UP.554. We consider the equation: $(1+iz)^{2n}=i\cdot(1+z^2)^n$, $n\geq 1$natural number and $i^2=-1$.

a. Prove that the complex number i is a solution of the equation for any $n\geq 1$.
b. Solve the equation in the case $n=1$ and in one the cases $n=2$ or $n=3$.
c. Find the solution of the equation in the general case $n\in\mathbf{N}^*$.

RMM- Number 37, Summer 2025, Adalbert Kovacs, Romania

Solutie.

a. Se verifica $(1+i\cdot i)^{2n}=i\cdot(1+i^2)^n \Leftrightarrow (1-1)^{2n}=i\cdot(1-1)^n \Leftrightarrow 0^{2n}=i\cdot 0^n \Leftrightarrow 0=0$.

b. Pentru $n=1$ avem ecuatia $(1+iz)^2=i\cdot(1+z^2) \Leftrightarrow (1+i)z^2-2iz+i-1=0$, cu $\Delta=4$.

Obtinem $z_1=2, z_2=2i$.

Pentru $n=2$ avem ecuatia $(1+iz)^4=i\cdot(1+z^2)^2 \Leftrightarrow (1-i)z^4-4iz^3-(6+2i)z^2+4iz+1-i=0$

$$\Leftrightarrow (z-i)^2(z^2+2z-1)=0.$$

Obtinem $z_1=i, z_{2,3}=-1\pm\sqrt{2}$.

Pentru $n=3$ avem ecuatia $(1+iz)^6=i\cdot(1+z^2)^3 \Leftrightarrow$

$$\Leftrightarrow (1+i)z^6-6iz^5+(3i-15)z^4+20iz^3+(3i+15)z^2-6iz+i-1=0$$

$$\Leftrightarrow (z-i)^3(z^3-3z^2-3z+1)=0 \Leftrightarrow (z-i)^3(z+1)(z^2-4z+1)=0.$$

Obtinem $z_1 = i, z_2 = -1 z_{3,4} = 2 \pm \sqrt{3}$.

c. Pentru rezolvarea ecuatiei $(1+iz)^{2n} = i \cdot (1+z^2)^n$ distingem cazurile:

i).Cazul1. $1+z^2 \neq 0$.

$(1+iz)^{2n} = i \cdot (1+z^2)^n \Leftrightarrow \left(\frac{1+2iz-z^2}{1+z^2}\right)^n = i, 1+z^2 \neq 0$.

Notand $w = \frac{1+2iz-z^2}{1+z^2}$ rezolvam ecuatia $w^n = i \Leftrightarrow w^n = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$, cu solutiile

$w_k = \cos\frac{\frac{\pi}{2}+2k\pi}{n} + i\sin\frac{\frac{\pi}{2}+2k\pi}{n}, k = \overline{0, n-1}$.

Revenind la notatie avem:

$w = \frac{1+2iz-z^2}{1+z^2} \Leftrightarrow (1+w)z^2 - iz + w - 1 = 0$, cu $\Delta = 3-4w^2 \Rightarrow z = i \pm \sqrt{3-4w^2}$,unde

$w = \cos\frac{\frac{\pi}{2}+2k\pi}{n} + i\sin\frac{\frac{\pi}{2}+2k\pi}{n}, k = \overline{0, n-1}$.

ii).Cazul2. $1+z^2 = 0$.

$1+z^2 = 0 \Leftrightarrow z^2 = -1 \Leftrightarrow z = \pm i$.

Problema este complet rezolvata.

Problema157.

SP.543. Let be $f:[0,1] \to [0,20], f(x) = 2\cdot 3^x + 4\cdot 5^x - 6$ .Find:

$$\Omega = \int_0^{20} f^{-1}(x)dx.$$

RMM- Number 37, Summer 2025, Daniel Sitaru, Romania

Solutie.

Folosim egalitatea lui Young:

Daca $f:[0,a] \to [0,b]$ este o functie derivabila, strict crescatoare cu $f(0) = 0$ si $f(a) = b$, atunci are loc egalitatea $\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx = ab$.

In cazul nostru $f:[0,1] \to [0,20], f(x) = 2\cdot 3^x + 4\cdot 5^x - 6$ este o functie derivabila, strict crescatoare(exponentiale cu baza supraunitara), cu $f(0) = 0$ si $f(1) = 20$.

Aplicand egalitatea lui Young obtinem:

$\int_0^1 f(x)dx + \int_0^{20} f^{-1}(x)dx = 20$.

Avem $\int_0^1 f(x)dx = \int_0^1 (2\cdot 3^x + 4\cdot 5^x - 6)dx = \left(2\cdot\frac{3^x}{\ln 3} + 4\cdot\frac{5^x}{\ln 5} - 6x\right)\Bigg|_0^1 = \left(2\cdot\frac{3}{\ln 3} + 4\cdot\frac{5}{\ln 5} - 6\right) -$

$-\left(2\cdot\frac{1}{\ln 3} + 4\cdot\frac{1}{\ln 5}\right) = \frac{4}{\ln 3} + \frac{16}{\ln 5} - 6$.

Obtinem:

$\Omega = \int_0^{20} f^{-1}(x)dx = 20 - \left(\frac{4}{\ln 3} + \frac{16}{\ln 5} - 6\right) = 26 - \frac{4}{\ln 3} - \frac{16}{\ln 5}$.

Remarca.

Let be $f:[0,1]\to[0,25]$, $f(x)=3\cdot 4^x+4\cdot 5^x-7$ .Find:

$$\Omega=\int_0^{25} f^{-1}(x)dx.$$

Marin Chirciu

Solutie.
Folosim egalitatea lui Young:
Daca $f:[0,a]\to[0,b]$ este o functie derivabila, strict crescatoare cu $f(0)=0$ si $f(a)=b$, atunci are loc egalitatea $\int_0^a f(x)dx+\int_0^b f^{-1}(x)dx=ab$.

In cazul nostru $f:[0,1]\to[0,25]$, $f(x)=3\cdot 4^x+4\cdot 5^x-7$ este o functie derivabila, strict crescatoare(exponentiale cu baza supraunitara), cu $f(0)=0$ si $f(1)=25$.
Aplicand egalitatea lui Young obtinem:

$$\int_0^1 f(x)dx+\int_0^{25} f^{-1}(x)dx=25.$$

Avem $\int_0^1 f(x)dx=\int_0^1(3\cdot 4^x+4\cdot 5^x-7)dx=\left(3\cdot\frac{4^x}{\ln 4}+4\cdot\frac{5^x}{\ln 5}-6x\right)\Big|_0^1=\left(3\cdot\frac{4}{\ln 4}+4\cdot\frac{5}{\ln 5}-7\right)-$
$-\left(3\cdot\frac{1}{\ln 4}+4\cdot\frac{1}{\ln 5}\right)=\frac{9}{\ln 3}+\frac{16}{\ln 5}-7$.
Obtinem:

$$\Omega=\int_0^{20} f^{-1}(x)dx=25-\left(\frac{9}{\ln 4}+\frac{16}{\ln 5}-7\right)=32-\frac{9}{\ln 4}-\frac{16}{\ln 5}.$$

Problema158.

UP.550. If $f:[0,1]\to\mathbf{R}$, f continous and $\int_0^1 xf(x)dx=a$, $\int_0^1 f(x)dx=b$, $a,b\in\mathbf{R}$ then:

$$\int_0^1 f^2(x)dx\ge 3(a-b)^2.$$

RMM- Number 37, Summer 2025, Daniel Sitaru, Romania

Solutie.
Folosim inegalitatea CBS sub forma integrala:

Daca $f,g:[a,b]\to\mathbf{R}$, f,g integrabile atunci $\int_a^b f^2(x)dx\int_a^b g^2(x)dx\ge\left(\int_a^b f(x)g(x)dx\right)^2$.

Punand $[a,b]=[0,1]$ si $g(x)=x-1$ obtinem:

$$\int_0^1 f^2(x)dx\int_0^1(x-1)^2\ge\left(\int_0^1 f(x)(x-1)dx\right)^2\Rightarrow\int_0^1 f^2(x)dx\cdot\frac{1}{3}\ge\left(\int_0^1 xf(x)dx-\int_0^1 f(x)dx\right)^2\Rightarrow$$
$$\Rightarrow\int_0^1 f^2(x)dx\cdot\frac{1}{3}\ge(a-b)^2\Rightarrow\int_0^1 f^2(x)dx\ge 3(a-b)^2.$$

Remarca.

If $f:[0,1]\to\mathbf{R}$, f continous and $\int_0^1 xf(x)dx=a$, $\int_0^1 f(x)dx=b$, $a,b\in\mathbf{R}$ then:

$$\int_0^1 f^2(x)dx \ge (3a-2b)^2.$$

Marin Chirciu

Solutie.
Folosim inegalitatea CBS sub forma integrala:

Daca $f,g:[a,b]\to\mathbf{R}$, f,g integrabile atunci $\int_a^b f^2(x)dx\int_a^b g^2(x)dx \ge \left(\int_a^b f(x)g(x)dx\right)^2$.

Punand $[a,b]=[0,1]$ si $g(x)=3x-2$ obtinem:

$$\int_0^1 f^2(x)dx\int_0^1 (3x-2)^2 \ge \left(\int_0^1 f(x)(3x-2)dx\right)^2 \Rightarrow \int_0^1 f^2(x)dx\cdot 1 \ge \left(3\int_0^1 xf(x)dx-2\int_0^1 f(x)dx\right)^2 \Rightarrow$$

$$\Rightarrow \int_0^1 f^2(x)dx \ge (3a-2b)^2.$$

Problema159.
SP.553. If $0\le a\le b<1$ then:

$$6\int_a^b \log\left(\frac{1+x}{1-x}\right)dx \ge \left(b^2-a^2\right)\left(b^2+a^2+6\right).$$

RMM- Number 37, Summer 2025, Daniel Sitaru, Romania

Solutie.
Aratam ca $\log\left(\frac{1+x}{1-x}\right)\ge 2x+\frac{2}{3}x^3, 0\le x<1$.

Intr-adevar:

Consideram functia $f(x)=\log\left(\frac{1+x}{1-x}\right)-2x-\frac{2}{3}x^3, 0\le x<1$.

Avem $f'(x)=\frac{2x^4}{1-x^2}\ge 0 \Rightarrow f$ este crescatoare pe $[0,1) \Rightarrow f(x)\ge f(0)=0 \Rightarrow$

$$\Rightarrow \log\left(\frac{1+x}{1-x}\right)-2x-\frac{2}{3}x^3\ge 0 \Rightarrow \log\left(\frac{1+x}{1-x}\right)\ge 2x+\frac{2}{3}x^3.$$

Folosind $\log\left(\frac{1+x}{1-x}\right)\ge 2x+\frac{2}{3}x^3$ obtinem:

$$6\int_a^b \log\left(\frac{1+x}{1-x}\right)dx \ge 6\int_a^b\left(2x+\frac{2}{3}x^3\right)dx = 6\left(x^2+\frac{x^4}{6}\right)\Bigg|_a^b = 6\left(b^2-a^2+\frac{b^4-a^4}{6}\right)=$$

$$\left(b^2-a^2\right)\left(b^2+a^2+6\right).$$

Remarca.
If $0\le a\le b$ then:

$$6\int_a^b \log(1+x)dx \ge (a-b)\left(a^2+ab+b^2-3a-3b\right).$$

Marin Chirciu

Problema160.
SP.551. Find:

$$\Omega=\lim_{n\to\infty} n\int_0^1 \frac{x^3\left(1+x^{4n-8}\right)}{\left(1+x^4\right)^n}dx.$$

RMM- Number 37, Summer 2025, Daniel Sitaru, Romania

Solutie.

Calculam $\int_0^1 \frac{x^3\left(1+x^{4n-8}\right)}{\left(1+x^4\right)^n}dx$.

Facem substitutia $x^4=t \Rightarrow 4x^3dx=dt$.

Obtinem

$$\int_0^1 \frac{x^3\left(1+x^{4n-8}\right)}{\left(1+x^4\right)^n}dx=\frac{1}{4}\int_0^1\frac{1+t^{n-2}}{(1+t)^n}dt=\frac{1}{4}\left(\int_0^1\frac{1}{(1+t)^n}dt+\int_0^1\frac{t^{n-2}}{(1+t)^n}dt\right)=\frac{1}{4}\left(\int_0^1(t+1)^{-n}dt+\int_0^1\left(\frac{t}{t+1}\right)^{n-2}\frac{1}{t^2}dt\right)=$$

$$=\frac{1}{4}\left(\frac{(t+1)^{-n+1}}{-n+1}\Bigg|_0^1+\int_0^1\left(\frac{t}{t+1}\right)^{n-2}\left(\frac{t}{t+1}\right)'dt\right)=\frac{1}{4}\left(\frac{2^{-n+1}}{-n+1}-\frac{1}{-n+1}+\frac{\left(\frac{t}{t+1}\right)^{n-2+1}}{n-2+1}\Bigg|_0^1\right)=$$

$$=\frac{1}{4}\left(\frac{2^{-n+1}}{-n+1}-\frac{1}{-n+1}+\frac{\left(\frac{1}{2}\right)^{n-1}}{n-1}-0\right)=\frac{1}{4}\left(\frac{2^{-n+1}}{-n+1}+\frac{1}{n-1}+\frac{2^{-n+1}}{n-1}\right)=\frac{1}{4}\cdot\frac{1}{n-1}=\frac{1}{4(n-1)}.$$

Rezulta $\int_0^1 \frac{x^3\left(1+x^{4n-8}\right)}{\left(1+x^4\right)^n}dx=\frac{1}{4(n-1)}$.

In final $\Omega=\lim_{n\to\infty} n\int_0^1 \frac{x^3\left(1+x^{4n-8}\right)}{\left(1+x^4\right)^n}dx=\lim_{n\to\infty} n\frac{1}{4(n-1)}=\frac{1}{4}$.

Remarca.

Let $k>1$ fixed. Find:

$$\Omega=\lim_{n\to\infty} n\int_0^1 \frac{x^{k-1}\left(1+x^{kn-2k}\right)}{\left(1+x^k\right)^n}dx.$$

Marin Chirciu

Solutie.

Calculam $\int_0^1 \frac{x^{k-1}\left(1+x^{kn-2k}\right)}{\left(1+x^k\right)^n}$.

Facem substitutia $x^k=t \Rightarrow kx^{k-1}dx=dt$.

Rezulta $\int_0^1 \frac{x^{k-1}\left(1+x^{kn-2k}\right)}{\left(1+x^k\right)^n}=\frac{1}{k(n-1)}$.

In final $\Omega=\lim_{n\to\infty} n\int_0^1 \frac{x^{k-1}\left(1+x^{kn-2k}\right)}{\left(1+x^k\right)^n}dx=\lim_{n\to\infty} n\frac{1}{k(n-1)}=\frac{1}{k}$.

Problema161.

JP.542. Solve for real numbers

$$\begin{cases} \dfrac{6x+6y}{9+4xy}=z \\ \dfrac{6y+6z}{9+4yz}=x \\ \dfrac{6z+6x}{9+4zx}=y \end{cases}.$$

RMM- Number 37, Summer 2025, Daniel Sitaru, Romania

Solutie.

$$\begin{cases} \dfrac{6x+6y}{9+4xy}=z \\ \dfrac{6y+6z}{9+4yz}=x \\ \dfrac{6z+6x}{9+4zx}=y \end{cases} \Leftrightarrow \begin{cases} 6x+6y=9z+4xyz \\ 6y+6z=9x+4xyz \\ 6z+6x=9y+4xyz \end{cases}.$$

Scad ecuatiile(1) si (2)$\Rightarrow 6(x-z)=9(z-x) \Rightarrow x=z$.

Scad ecuatiile(2) si (3)$\Rightarrow 6(y-x)=9(x-y) \Rightarrow x=y$.

Folosind ecuatia(1) si $x=y=z \Rightarrow 4x^3-3x=0 \Rightarrow x\in\left\{-\dfrac{\sqrt{3}}{2},0,\dfrac{\sqrt{3}}{2}\right\}$.

Multimea solutiilor sistemului este $S=\left\{\left(-\dfrac{\sqrt{3}}{2},-\dfrac{\sqrt{3}}{2},-\dfrac{\sqrt{3}}{2}\right),(0,0,0),\left(\dfrac{\sqrt{3}}{2},\dfrac{\sqrt{3}}{2},\dfrac{\sqrt{3}}{2}\right)\right\}$.

Remarca.

Let $0<\lambda\le 2n$ fixed. Solve for real numbers

$$\begin{cases} \dfrac{\lambda nx+\lambda ny}{\lambda^2+n^2xy}=z \\ \dfrac{\lambda ny+\lambda nz}{\lambda^2+n^2yz}=x \\ \dfrac{\lambda nz+\lambda nx}{\lambda^2+n^2zx}=y \end{cases}.$$

Marin Chirciu

Multimea solutiilor sistemului este :

$$S=\left\{\left(-\frac{\sqrt{2\lambda n-\lambda^2}}{n},-\frac{\sqrt{2\lambda n-\lambda^2}}{n},-\frac{\sqrt{2\lambda n-\lambda^2}}{n}\right),(0,0,0),\left(\frac{\sqrt{2\lambda n-\lambda^2}}{n},\frac{\sqrt{2\lambda n-\lambda^2}}{n},\frac{\sqrt{2\lambda n-\lambda^2}}{n}\right)\right\}$$

Problema162.

L271. Demonstrați că în orice triunghi are loc inegalitatea

1) $$\frac{bc}{(p-a)^2}+\frac{ca}{(p-b)^2}+\frac{ab}{(p-c)^2}\ge\frac{20R-4r}{3r}.$$

Recreații Matematice2/20214, Andy Brojbeanu, elev, Târgoviște

2) $$\frac{bc}{(p-a)^2}+\frac{ca}{(p-b)^2}+\frac{ab}{(p-c)^2}\ge\frac{nR}{r}+12-2n,\ \text{unde } -8\le n\le 8.$$

Marin Chirciu, Pitești

Soluție.
Avem

$$\sum\frac{bc}{(p-a)^2}=\frac{\sum bc(p-b)^2(p-c)^2}{\prod(p-a)^2}=\frac{r^3\left[p^2(r-8R)+(4R+r)^3\right]}{(r^2p)^2}=\frac{p^2(r-8R)+(4R+r)^3}{rp^2}.$$

Inegalitatea de demonstrat se scrie:

$\frac{p^2(r-8R)+(4R+r)^3}{rp^2}\geq\frac{nR}{r}+12-2n\Leftrightarrow(4R+r)^3\geq p^2\left[(n+8)R+(11-2n)r\right]$, care rezultă din inegalitatea Blundon-Gerretsen $p^2\leq\frac{R(4R+r)^2}{2(2R-r)}$ (adevărată din identitatea $H\Gamma^2=4R^2\left[1-\frac{2p^2(2R-r)}{R(4R+r)^2}\right]$, Γ este punctul lui Gergonne:intersecția dreptelor AA',BB',CC', unde A',B',C' sunt punctele de tangență ale cercului înscris în triunghiul ABC cu laturile BC,CA,AB) și condiția $n+8\geq0$, care asigură $(n+8)R+(11-2n)r>0$.

Rămâne să arătăm că:

$$(4R+r)^3\geq\frac{R(4R+r)^2}{2(2R-r)}\cdot\left[(n+8)R+(11-2n)r\right]\Leftrightarrow(8-n)R^2+(2n-15)Rr-2r^2\geq0\Leftrightarrow$$

$\Leftrightarrow(R-2r)\left[(8-n)R+r\right]\geq0$, evident din inegalitatea lui Euler $R\geq2r$ și condiția $8-n\geq0$.

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

3) $$\frac{bc}{(p-a)^2}+\frac{ca}{(p-b)^2}+\frac{ab}{(p-c)^2}\geq\frac{8R}{r}-4.$$

4) $$\frac{bc}{(p-a)^2}+\frac{ca}{(p-b)^2}+\frac{ab}{(p-c)^2}\geq\frac{8R}{r}-4\geq\frac{20R-4r}{3r}.$$

5) $$\frac{bc}{(p-a)^2}+\frac{ca}{(p-b)^2}+\frac{ab}{(p-c)^2}\geq\frac{8R}{r}-4\geq\frac{nR}{r}+12-2n,\text{ unde }-8\leq n\leq8.$$

Problema163.

In ΔABC

$$\sum h_a\sin^2\frac{A}{2}\sum\frac{1}{\sin^2A}\geq9r.$$

Mathematical Inequalities 5/2024, George Apostolopoulos, Greece

Soluție.

Lema.

In ΔABC

$$\sum h_a\sin^2\frac{A}{2}\sum\frac{1}{\sin^2A}=\frac{(4R+r)\left[p^2\left(p^2+2r^2-8Rr\right)+r^2(4R+r)^2\right]}{8Rrp^2}.$$

Demonstrație.

Folosim $\sum h_a\sin^2\frac{A}{2}=\frac{r(4R+r)}{2R}$ și $\sum\frac{1}{\sin^2A}=\frac{p^2\left(p^2+2r^2-8Rr\right)+r^2(4R+r)^2}{4r^2p^2}$.

Folosind **Lema** obținem:

$$\sum h_a\sin^2\frac{A}{2}\sum\frac{1}{\sin^2A}=\frac{(4R+r)\left[p^2\left(p^2+2r^2-8Rr\right)+r^2(4R+r)^2\right]}{8Rrp^2}=$$

$$=\frac{4R+r}{8Rr}\left[p^2+2r^2-8Rr+\frac{r^2(4R+r)^2}{p^2}\right]\overset{Gerretsen}{\geq}\frac{4R+r}{8Rr}\left[16Rr-5r^2+2r^2-8Rr+\frac{r^2(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=$$

$$=\frac{4R+r}{8Rr}\left[8Rr-3r^2+\frac{2(2R-r)r^2}{R}\right]=\frac{(4R+r)(8R^2+Rr-2r^2)}{8R^2}\overset{Euler}{\geq}4R+r\overset{Euler}{\geq}9r.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$4R+r\leq\sum h_a\sin^2\frac{A}{2}\sum\frac{1}{\sin^2 A}\leq\frac{R}{2r}(4R+r).$$

Soluție.

Lema.

In ΔABC

$$\sum h_a\sin^2\frac{A}{2}\sum\frac{1}{\sin^2 A}=\frac{(4R+r)\left[p^2(p^2+2r^2-8Rr)+r^2(4R+r)^2\right]}{8Rrp^2}.$$

Demonstrație.

Folosim $\sum h_a\sin^2\frac{A}{2}=\frac{r(4R+r)}{2R}$ și $\sum\frac{1}{\sin^2 A}=\frac{p^2(p^2+2r^2-8Rr)+r^2(4R+r)^2}{4r^2p^2}$.

Inegalitatea din dreapta.

$$\sum h_a\sin^2\frac{A}{2}\sum\frac{1}{\sin^2 A}=\frac{(4R+r)\left[p^2(p^2+2r^2-8Rr)+r^2(4R+r)^2\right]}{8Rrp^2}=$$

$$=\frac{4R+r}{8Rr}\left[p^2+2r^2-8Rr+\frac{r^2(4R+r)^2}{p^2}\right]\overset{Gerretsen}{\leq}\frac{4R+r}{8Rr}\left[4R^2+4Rr+3r^2+2r^2-8Rr+\frac{r^2(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{4R+r}{8Rr}\left[4R^2-4Rr+5r^2+r(R+r)\right]=\frac{(4R+r)(4R^2-3Rr+6r^2)}{8Rr}\overset{Euler}{\leq}\frac{R}{2r}(4R+r).$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Inegalitatea din stânga.

$$\sum h_a\sin^2\frac{A}{2}\sum\frac{1}{\sin^2 A}=\frac{(4R+r)\left[p^2(p^2+2r^2-8Rr)+r^2(4R+r)^2\right]}{8Rrp^2}=$$

$$=\frac{4R+r}{8Rr}\left[p^2+2r^2-8Rr+\frac{r^2(4R+r)^2}{p^2}\right]\overset{Gerretsen}{\geq}\frac{4R+r}{8Rr}\left[16Rr-5r^2+2r^2-8Rr+\frac{r^2(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=$$

$$=\frac{4R+r}{8Rr}\left[8Rr-3r^2+\frac{2(2R-r)r^2}{R}\right]=\frac{(4R+r)(8R^2+Rr-2r^2)}{8R^2}\overset{Euler}{\geq}4R+r\overset{Euler}{\geq}9r.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$\frac{27R}{2}\le\sum r_a\cos^2\frac{A}{2}\sum\frac{1}{\sin^2 A}\le\frac{27R^3}{8r^2}.$$

Soluție.

Lema.

In ΔABC

$$\sum r_a\cos^2\frac{A}{2}\sum\frac{1}{\sin^2 A}=\frac{p^2\left(p^2+2r^2-8Rr\right)+r^2\left(4R+r\right)^2}{8Rr^2}.$$

Demonstrație.

Folosim $\sum r_a\cos^2\frac{A}{2}=\frac{p^2}{2R}$ și $\sum\frac{1}{\sin^2 A}=\frac{p^2\left(p^2+2r^2-8Rr\right)+r^2\left(4R+r\right)^2}{4r^2p^2}$.

Inegalitatea din dreapta.

$$\sum r_a\cos^2\frac{A}{2}\sum\frac{1}{\sin^2 A}=\frac{p^2\left(p^2+2r^2-8Rr\right)+r^2\left(4R+r\right)^2}{8Rr^2}\overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{\left(4R^2+4Rr+3r^2\right)\left(4R^2+4Rr+3r^2+2r^2-8Rr\right)+r^2\left(4R+r\right)^2}{8Rr^2}=$$

$$=\frac{2\left(R^4+2R^2r^2+Rr^3+r^4\right)}{Rr^2}\overset{Euler}{\le}\frac{27R^3}{8r^2}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Inegalitatea din stânga.

$$\sum r_a\cos^2\frac{A}{2}\sum\frac{1}{\sin^2 A}=\frac{p^2\left(p^2+2r^2-8Rr\right)+r^2\left(4R+r\right)^2}{8Rr^2}\overset{Gerretsen}{\ge}$$

$$\overset{Gerretsen}{\ge}\frac{\left(16Rr-5r^2\right)\left(16Rr-5r^2+2r^2-8Rr\right)+r^2\left(4R+r\right)^2}{8Rr^2}=\frac{2\left(9R^2-5Rr+r^2\right)}{R}\overset{Euler}{\ge}\frac{27R}{2}.$$

Egalitatea are loc dacă și numai dacă triunghiul este echilateral.

Remarca.

In ΔABC

$$3\sum h_a\sin^2\frac{A}{2}\le\sum r_a\cos^2\frac{A}{2}.$$

Soluție.

Folosim $\sum h_a\sin^2\frac{A}{2}=\frac{r\left(4R+r\right)}{2R}$ și $\sum r_a\cos^2\frac{A}{2}=\frac{p^2}{2R}$.

Dezvoltări, Marin Chirciu

Bibliografie:

1. C.Năstăsescu, C.Niță, M.Brandiburu, D. Joița , Exerciții de algebră, Editura Didactică și Pedagogică, București, 1992.
2. O. Bottema, R. Z. Djordjević, R. R. Janić, D. S. Mitrinović și P. M. Vasić, „Geometric Inequalities”, Groningen 1969, Olanda .
3. Marin Chirciu, Inegalități trigonometrice, de la inițiere la performanță, Editura Paralela 45, Pitești, 2016.
4. Marin Chirciu, Inegalități cu linii importante în triunghi ,de la inițiere la performanță, Editura Paralela 45, Pitești, 2018.
5. Marin Chirciu, Inegalități cu laturi și cu raze în triunghi, de la inițiere la performanță, Editura Paralela 45, Pitești, 2017.
6. Gazeta Matematică 2024.
7. Crux Mathematicorum 2024.

8. Mathematical Reflections 2024.
9. RMM- Number 37, 2024.
10. Sclipirea Minții 2024.
11. Recreații Matematice 2024.
12. George Apostolopoulos, Greece , Mathematical Inequalites5/2024.
13. Kostas Geronikolas, Greece, Mathematical Inequalities5/2024.
14. Thay Hai Math, Vietnam 5/2024.
15. Nguyen Minh Tho,Vietnam, Mathematical Inequalities 5/2024.
16. Nguyen Hung Cuong, Vietnam, RMM 5/2024.
17. Amir Sofi, Kosovo, Mathematics(College and High School) 5/2024.
18. Victor Ailioaiei,Vaslui , E:16880, GM-4/24.
19. Nguyen Viet Hung,Vietnam, Mathematical Reflections 3/2024.
20. Witcher Ben , Mathematical Inequalities5/2024.
21. Adrian Talaşman, Huşi, E:16882, GM 4/2024.
22. Costel Anghel, Negreni, Olt, SGM-4/24, S:24.128.
23. Dorin Marghidanu, Corabia, Olt, Mathematical Inequalities5/2024.
24. Tran Anh Tuan, THCS 5/2024,Vietnam.
25. Zaza Mzhavanadze, Georgia, RMM 5/2024.
26. Kunihiko Chikaya, RMM5/2024.
27. Eldeniz Hesenov, Georgia, RMM 5/2020.
28. Cristian Amorăriței, Suceava, SGM-2/2024.
29. Alexandru Moscaliuc, Botoşani, SGM-2/2024.
30. Pham Van Tuyen,Vietnam, THCS 5/2024.
31. Elton Papanikolla, Math Olymp5/2024.
32. Bui Duc Cuong, Vietnam, RMM 5/2024.
33. Zaza Mzhavanadze, Georgia, RMM 5/2024.
34. Vasile Mircea Popa, Sibiu, Sclipirea Minții 33/2024.
35. Liliana Tomiță, Suceava, SGM 2/2024.
36. Anișoara Rotariu, Dorohoi, GM 2/2024.
37. Florin Rotaru, Focşani , Sclipirea Minții Nr33/2024.
38. Gheorghe Molea, Romania, RMM- Number 37, Summer 2025.
39. Adalbert Kovacs, Romania, RMM- Number 37, Summer 2025.
40. Titu Andreescu, USA, Mathematical Reflections 2024.
41. D.M.Bătinețu-Giurgiu, Romania, RMM-42, Autumn-2024.
42. Elton Papanikola, MathOlymp 2/2024.
43. Gheorghe Crăciun, Ploiești, SGM-4/2024.
44. Neculai Stanciu, Romania, RMM5/2024.
45. Daniel Sitaru, Romania, RMM- Number 37, Summer 2025.
46. Andy Brojbeanu, Târgoviște, Recreații Matematice2/20214.
47. Nguyen Viet Hung,Vietnam, Mathematical Reflections6/2023.
48. Marin Chirciu, Inegalități algebrice 2, de la inițiere la performanță, Editura Paralela 45, Pitești, 2021.
49. Marin Chirciu, Inegalități geometrice 2, de la inițiere la performanță, Editura Paralela 45, Pitești, 2021.

Art 5800
1 Iunie 2024